

# Maximal Function Characterizations of Variable Hardy Spaces Associated with Non-negative Self-adjoint Operators Satisfying Gaussian Estimates

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**Abstract** Let  $p(\cdot) : \mathbb{R}^n \rightarrow (0, 1]$  be a variable exponent function satisfying the globally log-Hölder continuous condition and  $L$  a non-negative self-adjoint operator on  $L^2(\mathbb{R}^n)$  whose heat kernels satisfying the Gaussian upper bound estimates. Let  $H_L^{p(\cdot)}(\mathbb{R}^n)$  be the variable exponent Hardy space defined via the Lusin area function associated with the heat kernels  $\{e^{-t^2 L}\}_{t \in (0, \infty)}$ . In this article, the authors first establish the atomic characterization of  $H_L^{p(\cdot)}(\mathbb{R}^n)$ ; using this, the authors then obtain its non-tangential maximal function characterization which, when  $p(\cdot)$  is a constant in  $(0, 1]$ , coincides with a recent result by Song and Yan [Adv. Math. 287 (2016), 463-484] and further induces the radial maximal function characterization of  $H_L^{p(\cdot)}(\mathbb{R}^n)$  under an additional assumption that the heat kernels of  $L$  have the Hölder regularity.

## 1 Introduction

The main purpose of this article is to establish the non-tangential or radial maximal function characterizations of the Hardy space  $H_L^{p(\cdot)}(\mathbb{R}^n)$  introduced in [48]. Recall that the theory of classical Hardy spaces on the Euclidean space  $\mathbb{R}^n$  was introduced and developed in the 1960s and 1970s. Precisely, the real-variable theory of Hardy spaces on  $\mathbb{R}^n$  was initiated by Stein and Weiss [42] and then systematically developed by Fefferman and Stein [24], which has played an important role in modern harmonic analysis and been widely used in partial differential equations (see, for example, [16, 24, 41]). As was well known, the classical Hardy space is intimately connected with the Laplace operator  $\Delta := -\sum_{i=1}^n \partial_{x_i}^2$  on  $\mathbb{R}^n$ . Indeed, for  $p \in (0, 1]$ , the Hardy space  $H^p(\mathbb{R}^n)$  consists of all  $f \in \mathcal{S}'(\mathbb{R}^n)$  (the set of all *tempered distributions*) such that the area integral function

$$S(f)(\cdot) := \left\{ \int_0^\infty \int_{|y-\cdot|<t} \left| t^2 \Delta e^{-t^2 \Delta}(f)(y) \right|^2 \frac{dy dt}{t^{n+1}} \right\}^{\frac{1}{2}}$$

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belongs to  $L^p(\mathbb{R}^n)$ . Moreover, for  $p \in (0, 1]$ , the Hardy space  $H^p(\mathbb{R}^n)$  involves several different equivalent characterizations, for example, if  $f \in \mathcal{S}'(\mathbb{R}^n)$ , then

$$\begin{aligned} f \in H^p(\mathbb{R}^n) &\iff \sup_{t \in (0, \infty)} \left| e^{-t^2 \Delta}(f) \right| \in L^p(\mathbb{R}^n) \\ &\iff \sup_{t \in (0, \infty), |y - \cdot| < t} \left| e^{-t^2 \Delta}(f)(y) \right| \in L^p(\mathbb{R}^n). \end{aligned}$$

Also, it is well known that the Hardy space  $H^p(\mathbb{R}^n)$ , with  $p \in (0, 1]$ , is a suitable substitute of the Lebesgue space  $L^p(\mathbb{R}^n)$ , for example, the classical Riesz transform is bounded on  $H^p(\mathbb{R}^n)$ , but not on  $L^p(\mathbb{R}^n)$  when  $p \in (0, 1]$ . However, in many situations, the standard theory of Hardy spaces is not applicable, for example, the Riesz transform  $\nabla L^{-1/2}$  may not be bounded from the Hardy space  $H^1(\mathbb{R}^n)$  to  $L^1(\mathbb{R}^n)$  when  $L$  is a second-order divergence form elliptic operator with complex bounded measurable coefficients (see [29]). Motivated by this, the topic for developing a real-variable theory of Hardy spaces that are adapted to different differential operators has inspired great interests in the last decade and has become a very active research topic in harmonic analysis (see, for example, [3, 6, 21, 22, 23, 28, 29, 30, 32, 44, 45, 48]).

Particularly, let  $L$  be a linear operator on  $L^2(\mathbb{R}^n)$  and generate an analytic semigroup  $\{e^{-tL}\}_{t>0}$  with heat kernels having pointwise upper bounds. Then, by using the Lusin area function associated with these heat kernels, Auscher, Duong and McIntosh [3] initially studied the Hardy space  $H_L^1(\mathbb{R}^n)$  associated with the operator  $L$ . Based on this, Duong and Yan [21, 22] introduced the BMO-type space  $\text{BMO}_L(\mathbb{R}^n)$  associated with  $L$  and proved that the dual space of  $H_L^1(\mathbb{R}^n)$  is just  $\text{BMO}_{L^*}(\mathbb{R}^n)$ , where  $L^*$  denotes the *adjoint operator* of  $L$  in  $L^2(\mathbb{R}^n)$ . Later, Yan [44] further generalized these results to the Hardy spaces  $H_L^p(\mathbb{R}^n)$  with  $p$  close to, but less than, 1 and, more generally, the Orlicz-Hardy space associated with such operator was investigated by Jiang et al. [32]. Very recently, under the assumption that  $L$  is a non-negative self-adjoint operator whose heat kernels satisfying Gaussian upper bound estimates, Song and Yan [40] established a characterization of Hardy spaces  $H_L^p(\mathbb{R}^n)$  via the non-tangential maximal function associated with the heat semigroup of  $L$  based on a subtle modification of technique due to Calderón [8], which was further generalized into the Musielak-Orlicz-Hardy space in [46].

Another research direction of generalized Hardy spaces is the variable exponent Hardy space, which also extends the variable Lebesgue space. Recall that the variable Lebesgue space  $L^{p(\cdot)}(\mathbb{R}^n)$ , with a variable exponent  $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ , consists of all measurable functions  $f$  such that  $\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx < \infty$ . The study of variable Lebesgue spaces can be traced back to Birnbaum-Orlicz [5] and Orlicz [36], but the modern development started with the article [33] of Kováčik and Rákosník as well as [12] of Cruz-Uribe and [17] of Diening, and nowadays have been widely used in harmonic analysis (see, for example, [13, 18]). Moreover, variable function spaces also have interesting applications in fluid dynamics [1, 37], image processing [10], partial differential equations and variational calculus [2, 27, 38]. Recall that the variable exponent Hardy space  $H^{p(\cdot)}(\mathbb{R}^n)$  was introduced by Nakai and Sawano [35] and, independently, by Cruz-Uribe and Wang [15] with some weaker assumptions on  $p(\cdot)$  than those used in [35], which was further investigated by Sawano [39], Zhuo et al. [50] and Yang et al. [49].

Let  $p(\cdot) : \mathbb{R}^n \rightarrow (0, 1]$  be a variable exponent function satisfying the globally log-Hölder continuous condition. Very recently, the authors [48] introduced the Hardy space  $H_L^{p(\cdot)}(\mathbb{R}^n)$  via the Lusin area function associated with a linear operator  $L$  on  $L^2(\mathbb{R}^n)$  whose heat kernels having pointwise upper bound, and obtained their molecular characterizations. In this article, we aim at establishing equivalent characterizations of  $H_L^{p(\cdot)}(\mathbb{R}^n)$ , under the additional assumption that  $L$  is a non-negative self-adjoint operator, in terms of maximal functions, including (grand) non-tangential maximal functions and (grand) radial maximal function. To this end, we first introduce the space  $H_{L,at,M}^{p(\cdot),q}(\mathbb{R}^n)$ , the variable atomic Hardy space associated with the operator  $L$  (see Definition 1.6 below), and then prove that  $H_L^{p(\cdot)}(\mathbb{R}^n)$  and  $H_{L,at,M}^{p(\cdot),q}(\mathbb{R}^n)$  coincide with equivalent quasi-norms (see Theorem 1.8 below). Based on the results from Song and Yan [40], we characterize  $H_{L,at,M}^{p(\cdot),q}(\mathbb{R}^n)$  via (grand) non-tangential maximal functions in Theorem 1.11 below, from which, we further deduce the (grand) radial maximal function characterizations of  $H_{L,at,M}^{p(\cdot),q}(\mathbb{R}^n)$  in Theorem 1.17 below under an additional Hölder continuous assumption on the heat kernels of  $L$  (see (1.7) below). Using Theorems 1.8, 1.11 and 1.17, we also obtain the corresponding characterizations of  $H_L^{p(\cdot)}(\mathbb{R}^n)$ , respectively, in terms of atoms, the (grand) non-tangential maximal functions and the (grand) radial maximal functions (see Corollary 1.18 below).

To state the results of this article, we begin with some notation and notions. A measurable function  $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$  is called a *variable exponent*. For any variable exponent  $p(\cdot)$ , let

$$(1.1) \quad p_- := \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x) \quad \text{and} \quad p_+ := \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x).$$

Denote by  $\mathcal{P}(\mathbb{R}^n)$  the collection of all variable exponents  $p(\cdot)$  satisfying  $0 < p_- \leq p_+ < \infty$ .

For a given variable exponent  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ , the *modular*  $\varrho_{p(\cdot)}$ , associated with  $p(\cdot)$ , is defined by setting  $\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx$  for any measurable function  $f$  and the *Luxemburg (quasi-)norm* of  $f$  is given by

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf \{ \lambda \in (0, \infty) : \varrho_{p(\cdot)}(f/\lambda) \leq 1 \}.$$

Then the *variable exponent Lebesgue space*  $L^{p(\cdot)}(\mathbb{R}^n)$  is defined to be the set of all measurable functions  $f$  such that  $\varrho_{p(\cdot)}(f) < \infty$ , equipped with the quasi-norm  $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ . For more properties on the variable exponent Lebesgue spaces, we refer the reader to [13, 18].

**Remark 1.1.** Let  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ . Then, for all  $f, g \in L^{p(\cdot)}(\mathbb{R}^n)$ ,

$$\|f + g\|_{L^{p(\cdot)}(\mathbb{R}^n)}^p \leq \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}^p + \|g\|_{L^{p(\cdot)}(\mathbb{R}^n)}^p,$$

where  $p := \min\{1, p_-\}$ , and, for all  $\lambda \in \mathbb{C}$ ,  $\|\lambda f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = |\lambda| \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ . In particular, when  $p_- \in [1, \infty)$ ,  $L^{p(\cdot)}(\mathbb{R}^n)$  is a Banach space (see [18, Theorem 3.2.7]).

In the present article, we always assume that the variable exponent  $p(\cdot)$  satisfies the *globally log-Hölder continuous condition*. Recall that a measurable function  $p(\cdot)$  is said to

satisfy the globally log-Hölder continuous condition, denoted by  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ , if there exists a positive constant  $C_{\log}(p)$  such that, for all  $x, y \in \mathbb{R}^n$ ,

$$|p(x) - p(y)| \leq \frac{C_{\log}(p)}{\log(e + 1/|x - y|)},$$

and there exist a positive constant  $C_{\infty}$  and a constant  $p_{\infty} \in \mathbb{R}$  such that, for all  $x \in \mathbb{R}^n$ ,

$$|p(x) - p_{\infty}| \leq \frac{C_{\infty}}{\log(e + |x|)}.$$

In what follows, for any  $r \in (0, \infty)$  and measurable set  $E \subset \mathbb{R}^n$ , denote by  $L^r(E)$  the set of all measurable functions  $f$  such that  $\|f\|_{L^r(E)} := \{\int_E |f(x)|^r dx\}^{1/r} < \infty$ .

In this article, unless otherwise stated, we always assume that  $L$  is a densely defined linear operator on  $L^2(\mathbb{R}^n)$  and satisfies the following assumptions:

**Assumption 1.2.**  $L$  is non-negative and self-adjoint;

**Assumption 1.3.** The kernels of the semigroup  $\{e^{-tL}\}_{t>0}$ , denoted by  $\{K_t\}_{t>0}$ , are measurable functions on  $\mathbb{R}^n \times \mathbb{R}^n$  and satisfy the Gaussian upper bound estimates, namely, there exist positive constants  $C$  and  $c$  such that, for all  $t \in (0, \infty)$  and  $x, y \in \mathbb{R}^n$ ,

$$|K_t(x, y)| \leq \frac{C}{t^{n/2}} \exp\left\{-\frac{|x - y|^2}{ct}\right\}.$$

**Remark 1.4.** (i) One of the typical example of operators  $L$  satisfying both Assumptions (1.2) and 1.3 is the Schrödinger operator  $L := -\Delta + V$  with  $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$ .

(ii) If  $\{e^{-tL}\}_{t>0}$  is a bounded analytic semigroup on  $L^2(\mathbb{R}^n)$  whose kernels  $\{K_t\}_{t>0}$  satisfy Assumptions 1.2 and 1.3, then, for any  $j \in \mathbb{N} := \{1, 2, \dots\}$ , there exists a positive constant  $C$  such that, for all  $t \in (0, \infty)$  and almost every  $x, y \in \mathbb{R}^n$ ,

$$(1.2) \quad \left| t^j \frac{\partial^j K_t(x, y)}{\partial t^j} \right| \leq \frac{C}{t^{n/2}} \exp\left\{-\frac{|x - y|^2}{ct}\right\};$$

see, for example, [44, p. 4386].

For all functions  $f \in L^2(\mathbb{R}^n)$ , define the *Lusin area function*  $S_L(f)$  associated with the operator  $L$  by setting, for all  $x \in \mathbb{R}^n$ ,

$$S_L(f)(x) := \left\{ \int_{\Gamma(x)} \left| t^2 L e^{-t^2 L}(f)(y) \right|^2 \frac{dy dt}{t^{n+1}} \right\}^{1/2},$$

here and hereafter, for all  $x \in \mathbb{R}^n$ ,

$$(1.3) \quad \Gamma(x) := \{(y, t) \in \mathbb{R}^n \times (0, \infty) : |y - x| < t\}.$$

In [3], Auscher et al. proved that, for any  $p \in (1, \infty)$ , there exists a positive constant  $C$  such that, for all  $f \in L^p(\mathbb{R}^n)$ ,

$$(1.4) \quad C^{-1} \|f\|_{L^p(\mathbb{R}^n)} \leq \|S_L(f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)};$$

see also Duong and McIntosh [20] and Yan [43].

We now recall the definition of the variable exponent Hardy space associated with operator, which was first studied in [48].

**Definition 1.5.** Let  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$  with  $p_+ \in (0, 1]$  and  $L$  be an operator satisfying Assumptions 1.2 and 1.3. A function  $f \in L^2(\mathbb{R}^n)$  is said to be in  $\mathbb{H}_L^{p(\cdot)}(\mathbb{R}^n)$  if  $S_L(f) \in L^{p(\cdot)}(\mathbb{R}^n)$ ; moreover, define

$$\|f\|_{H_L^{p(\cdot)}(\mathbb{R}^n)} := \|S_L(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \left[ \frac{S_L(f)(x)}{\lambda} \right]^{p(x)} dx \leq 1 \right\}.$$

Then the *variable Hardy space associated with operator  $L$* , denoted by  $H_L^{p(\cdot)}(\mathbb{R}^n)$ , is defined to be the completion of  $\mathbb{H}_L^{p(\cdot)}(\mathbb{R}^n)$  in the quasi-norm  $\|\cdot\|_{H_L^{p(\cdot)}(\mathbb{R}^n)}$ .

Next we introduce the notions of the  $(p(\cdot), q, M)_L$ -atom and the atomic variable exponent Hardy space  $H_{L,at,M}^{p(\cdot),q}(\mathbb{R}^n)$ .

**Definition 1.6.** Let  $L$  and  $p(\cdot)$  be as in Definition 1.5,  $q \in (1, \infty]$  and  $M \in \mathbb{N}$ .

(I) Let  $\mathcal{D}(L^M)$  be the domain of  $L^M$  and  $Q \subset \mathbb{R}^n$  a cube. A function  $\alpha \in L^q(\mathbb{R}^n)$  is called a  $(p(\cdot), q, M)_L$ -atom associated with the cube  $Q$  if there exists a function  $b \in \mathcal{D}(L^M)$  such that

- (i)  $\alpha = L^M b$  and, for all  $j \in \{0, 1, \dots, M\}$ ,  $\text{supp}(L^j b) \subset Q$ ;
- (ii) for all  $j \in \{0, 1, \dots, M\}$ ,  $\|([\ell(Q)]^2 L)^j b\|_{L^q(\mathbb{R}^n)} \leq [\ell(Q)]^{2M} |Q|^{1/q} \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}$ , where  $\ell(Q)$  denotes the *side length* of  $Q$ .

(II) Let  $f \in L^2(\mathbb{R}^n)$ . Then

$$(1.5) \quad f = \sum_{j \in \mathbb{N}} \lambda_j \alpha_j$$

is called an atomic  $(p(\cdot), q, M)_L$ -representation of  $f$  if the sequences  $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$  and  $\{\alpha_j\}_{j \in \mathbb{N}}$  are  $(p(\cdot), q, M)_L$ -atoms associated with cubes  $\{Q_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^n$  such that (1.5) converges in  $L^2(\mathbb{R}^n)$  and

$$\int_{\mathbb{R}^n} \left\{ \sum_{j \in \mathbb{N}} \left[ \frac{|\lambda_j| \chi_{Q_j}(x)}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{p_-} \right\}^{p(x)/p_-} dx < \infty.$$

Let

$$\mathbb{H}_{L,at,M}^{p(\cdot),q}(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) : f \text{ has an atomic } (p(\cdot), q, M)_L\text{-representation}\}$$

equipped with the quasi-norm  $\|f\|_{H_{L,at,M}^{p(\cdot),q}(\mathbb{R}^n)}$  given by

$$\inf \left\{ \mathcal{B}(\{\lambda_j \alpha_j\}_{j \in \mathbb{N}}) : \sum_{j \in \mathbb{N}} \lambda_j \alpha_j \text{ is an atomic } (p(\cdot), q, M)_L\text{-representation of } f \right\},$$

where

$$\mathcal{B}(\{\lambda_j \alpha_j\}_{j \in \mathbb{N}}) := \left\| \left\{ \sum_{j \in \mathbb{N}} \left[ \frac{|\lambda_j| \chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{p_-} \right\}^{1/p_-} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

and the infimum is taken over all the atomic  $(p(\cdot), q, M)_L$ -representations of  $f$  as above.

The *atomic variable exponent Hardy space*  $H_{L, \text{at}, M}^{p(\cdot), q}(\mathbb{R}^n)$  is then defined to be the completion of the set  $\mathbb{H}_{L, \text{at}, M}^{p(\cdot), q}(\mathbb{R}^n)$  with respect to the quasi-norm  $\|\cdot\|_{H_{L, \text{at}, M}^{p(\cdot), q}(\mathbb{R}^n)}$ .

**Remark 1.7.** It is easy to see that, for any  $q \in (1, \infty)$  and  $M \in \mathbb{N}$ ,

$$H_{L, \text{at}, M}^{p(\cdot), \infty}(\mathbb{R}^n) \subset H_{L, \text{at}, M}^{p(\cdot), q}(\mathbb{R}^n).$$

The first main result of this article is stated as follows, which, in the case that  $p(\cdot) \equiv \text{constant} \in (0, 1]$ , was established in [19, 28] (see also [31]).

**Theorem 1.8.** *Let  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$  with  $p_+ \in (0, 1]$ ,  $q \in (1, \infty]$ ,  $M \in (\frac{n}{2}[\frac{1}{p_-} - 1], \infty) \cap \mathbb{N}$  and  $L$  be a linear operator on  $L^2(\mathbb{R}^n)$  satisfying Assumptions 1.2 and 1.3. Then  $H_{L, \text{at}, M}^{p(\cdot), q}(\mathbb{R}^n)$  and  $H_L^{p(\cdot)}(\mathbb{R}^n)$  coincide with equivalent quasi-norms.*

In this article, we use  $\mathcal{S}(\mathbb{R}^n)$  to denote the space of all Schwartz functions on  $\mathbb{R}^n$ .

**Definition 1.9.** (i) Let  $\phi \in \mathcal{S}(\mathbb{R})$  be an even function with  $\phi(0) = 1$ . For any  $a \in (0, \infty)$  and  $f \in L^2(\mathbb{R}^n)$ , the *non-tangential maximal function* of  $f$  is defined by setting, for all  $x \in \mathbb{R}^n$ ,

$$\phi_{L, \nabla, a}^*(f)(x) := \sup_{t \in (0, \infty), |y-x| < at} \left| \phi(t\sqrt{L})(f)(y) \right|.$$

A function  $f \in L^2(\mathbb{R}^n)$  is said to be in the set  $\mathbb{H}_{L, \max}^{p(\cdot), \phi, a}(\mathbb{R}^n)$  if  $\phi_{L, \nabla, a}^*(f) \in L^{p(\cdot)}(\mathbb{R}^n)$ ; moreover, define  $\|f\|_{H_{L, \max}^{p(\cdot), \phi, a}(\mathbb{R}^n)} := \|\phi_{L, \nabla, a}^*(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ . Then the *variable exponent Hardy space*  $H_{L, \max}^{p(\cdot), \phi, a}(\mathbb{R}^n)$  is defined to be the completion of  $\mathbb{H}_{L, \max}^{p(\cdot), \phi, a}(\mathbb{R}^n)$  with respect to the quasi-norm  $\|\cdot\|_{H_{L, \max}^{p(\cdot), \phi, a}(\mathbb{R}^n)}$ .

Particularly, when  $\phi(x) := e^{-x^2}$  for all  $x \in \mathbb{R}^n$ , we use  $f_{L, \nabla}^*$  to denote  $\phi_{L, \nabla, 1}^*(f)$  and, in this case, denote the space  $H_{L, \max}^{p(\cdot), \phi, a}(\mathbb{R}^n)$  simply by  $H_{L, \max}^{p(\cdot)}(\mathbb{R}^n)$ .

(ii) For any  $f \in L^2(\mathbb{R}^n)$ , define the *grand non-tangential maximal function* of  $f$  by setting, for all  $x \in \mathbb{R}^n$ ,

$$\mathcal{G}_{L, \nabla}^*(f)(x) := \sup_{\phi \in \mathcal{F}(\mathbb{R})} \phi_{L, \nabla, 1}^*(f)(x),$$

where  $\mathcal{F}(\mathbb{R})$  denotes the set of all even functions  $\phi \in \mathcal{S}(\mathbb{R})$  satisfying  $\phi(0) \neq 0$  and

$$\sum_{k=0}^N \int_{\mathbb{R}} (1 + |x|)^N \left| \frac{d^k \phi(x)}{dx^k} \right|^2 dx \leq 1$$

with  $N$  being a large enough number depending on  $p(\cdot)$  and  $n$ . Then the *variable exponent Hardy space*  $H_{L, \max}^{p(\cdot), \mathcal{F}}(\mathbb{R}^n)$  is defined in the same way as  $H_{L, \max}^{p(\cdot), \phi, a}(\mathbb{R}^n)$  but with  $\phi_{L, \nabla, a}^*(f)$  replaced by  $\mathcal{G}_{L, \nabla}^*(f)$ .

**Remark 1.10.** By Assumption 1.3, we conclude that there exists a positive constant  $C$  such that, for any  $f \in L^2(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ ,  $f_{L,\nabla}^*(x) \leq C\mathcal{M}(f)(x)$ . Here and hereafter,  $\mathcal{M}$  denotes the *Hardy-Littlewood maximal operator*, which is defined by setting, for all locally integrable function  $f$  and  $x \in \mathbb{R}^n$ ,

$$\mathcal{M}(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

where the supremum is taken over all balls  $B$  of  $\mathbb{R}^n$ .

The second main result of this article is presented as follows.

**Theorem 1.11.** *Let  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$  with  $p_+ \in (0, 1]$ ,  $q \in (1, \infty]$ ,  $M \in (\frac{n}{2}[\frac{1}{p_-} - 1], \infty)$  and  $L$  be an operator satisfying Assumptions 1.2 and 1.3. Then, for any  $a \in (0, \infty)$  and  $\phi$  as in Definition 1.9, the spaces  $H_{L,\text{at},M}^{p(\cdot),q}(\mathbb{R}^n)$ ,  $H_{L,\text{max}}^{p(\cdot),\mathcal{F}}(\mathbb{R}^n)$  and  $H_{L,\text{max}}^{p(\cdot),\phi,a}(\mathbb{R}^n)$  coincide with equivalent quasi-norms.*

**Remark 1.12.** When  $p(\cdot) \equiv \text{constant} \in (0, 1]$ , the conclusion of Theorem 1.11 was proved by Song and Yan in [40, Theorem 1.4].

**Definition 1.13.** (i) Let  $\phi \in \mathcal{S}(\mathbb{R})$  be an even function with  $\phi(0) = 1$ . For  $f \in L^2(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ , let

$$\phi_{L,+}^*(f)(x) := \sup_{t \in (0, \infty)} \left| \phi(t\sqrt{L})(f)(x) \right|.$$

Particularly, when  $\phi(x) := e^{-x^2}$  for all  $x \in \mathbb{R}^n$ , we use  $f_{L,+}^*$  to denote  $\phi_{L,+}^*(f)$ . The *variable exponent Hardy space*  $H_{L,\text{rad}}^{p(\cdot)}(\mathbb{R}^n)$  is defined in the same way as  $H_{L,\text{max}}^{p(\cdot),\phi,a}(\mathbb{R}^n)$  but with  $\phi_{L,\nabla,a}^*(f)$  replaced by  $f_{L,+}^*$ .

(ii) For any  $f \in L^2(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ , let

$$\mathcal{G}_{L,+}^*(f)(x) := \sup_{\phi \in \mathcal{A}(\mathbb{R})} \phi_{L,+}^*(f)(x).$$

The *variable exponent Hardy space*  $H_{L,\text{rad}}^{p(\cdot),\mathcal{F}}(\mathbb{R}^n)$  is defined in the same way as  $H_{L,\text{max}}^{p(\cdot),\phi,a}(\mathbb{R}^n)$  but with  $\phi_{L,\nabla,a}^*(f)$  replaced by  $\mathcal{G}_{L,+}^*(f)$ .

**Remark 1.14.** We point out that, for any  $q \in (1, \infty]$  and  $M \in \mathbb{N}$ , the sets

$$H_{L,\text{at},M}^{p(\cdot),q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), H_{L,\text{max}}^{p(\cdot),\phi,a}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), H_{L,\text{max}}^{p(\cdot),\mathcal{F}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$$

and

$$H_{L,\text{rad}}^{p(\cdot),\phi,a}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), H_{L,\text{rad}}^{p(\cdot),\mathcal{F}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$$

are, respectively, dense in the spaces  $H_{L,\text{at},M}^{p(\cdot),q}(\mathbb{R}^n)$ ,  $H_{L,\text{max}}^{p(\cdot),\phi,a}(\mathbb{R}^n)$ ,  $H_{L,\text{max}}^{p(\cdot),\mathcal{F}}(\mathbb{R}^n)$ ,  $H_{L,\text{rad}}^{p(\cdot),\phi,a}(\mathbb{R}^n)$  and  $H_{L,\text{rad}}^{p(\cdot),\mathcal{F}}(\mathbb{R}^n)$ .



By the definitions of  $H_{L,\max}^{p(\cdot)}(\mathbb{R}^n)$  and  $H_{L,\text{rad}}^{p(\cdot)}(\mathbb{R}^n)$ , we easily know that the continuous inclusion  $H_{L,\max}^{p(\cdot)}(\mathbb{R}^n) \subset H_{L,\text{rad}}^{p(\cdot)}(\mathbb{R}^n)$  holds true. It is a natural question whether or not the continuous inclusion

$$(1.6) \quad H_{L,\text{rad}}^{p(\cdot)}(\mathbb{R}^n) \subset H_{L,\max}^{p(\cdot)}(\mathbb{R}^n)$$

holds true. We remark that, in the case of  $p(\cdot) \equiv \text{constant} \in (0, 1]$ , (1.6) has been proved in [46, Theorem 1.9] under the following additional Assumption 1.15 on the operator  $L$ , which gives an affirmative answer to the open question stated in [40, Remark 3.4].

**Assumption 1.15.** There exist positive constants  $C$  and  $\mu \in (0, 1]$  such that, for all  $t \in (0, \infty)$  and  $x, y_1, y_2 \in \mathbb{R}^n$ ,

$$(1.7) \quad |K_t(y_1, x) - K_t(y_2, x)| \leq \frac{C}{t^{n/2}} \frac{|y_1 - y_2|^\mu}{t^{\mu/2}}.$$

**Remark 1.16.** There exist some operators on  $\mathbb{R}^n$  whose heat kernels satisfy Assumption 1.15. These operators include Schrödinger operators with non-negative potentials belonging to the reverse Hölder class (see, for example, [23]) and second-order divergence form elliptic operators with bounded measurable real coefficients (see, for example, [4]).

Motivated by [46, Theorem 1.9], in this article, we also establish the following radial maximal function characterization of  $H_L^{p(\cdot)}(\mathbb{R}^n)$  via showing that (1.6) holds true.

**Theorem 1.17.** Let  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$  with  $p_+ \in (0, 1]$  and  $L$  be a linear operator on  $L^2(\mathbb{R}^n)$  satisfying Assumptions 1.2, 1.3 and 1.15. If  $q \in (1, \infty]$ ,  $M \in (\frac{n}{2}[\frac{1}{p_-} - 1], \infty) \cap \mathbb{N}$ , then the spaces  $H_{L,\text{at},M}^{p(\cdot),q}(\mathbb{R}^n)$ ,  $H_{L,\max}^{p(\cdot)}(\mathbb{R}^n)$  and  $H_{L,\text{rad}}^{p(\cdot)}(\mathbb{R}^n)$  coincide with equivalent quasi-norms.

As an immediate consequence of Theorems 1.8, 1.11 and 1.17, we have the following conclusion.

**Corollary 1.18.** Let  $p(\cdot)$ ,  $L$ ,  $q$  and  $M$  be as in Theorem 1.17. Then, for any  $a \in (0, \infty)$  and  $\phi$  being as in Definition 1.9, the spaces  $H_L^{p(\cdot)}(\mathbb{R}^n)$ ,  $H_{L,\text{at},M}^{p(\cdot),q}(\mathbb{R}^n)$ ,  $H_{L,\max}^{p(\cdot),\phi,a}(\mathbb{R}^n)$ ,  $H_{L,\max}^{p(\cdot),\mathcal{F}}(\mathbb{R}^n)$ ,  $H_{L,\text{rad}}^{p(\cdot)}(\mathbb{R}^n)$  and  $H_{L,\text{rad}}^{p(\cdot),\mathcal{F}}(\mathbb{R}^n)$  coincide with equivalent quasi-norms.

**Remark 1.19.** Let  $\varphi : \mathbb{R}^n \times [0, \infty) \rightarrow [0, \infty)$  be a growth function in [34]. D. Yang and S. Yang [46] established several maximal function characterizations of  $H_{\varphi,L}(\mathbb{R}^n)$ , the Musielak-Orlicz-Hardy spaces associated with operators  $L$  satisfying the same assumptions as in the article. Recall that the Musielak-Orlicz space  $L^\varphi(\mathbb{R}^n)$  is defined to be the set of all measurable functions  $f$  on  $\mathbb{R}^n$  such that

$$\|f\|_{L^\varphi(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \varphi(x, |f(x)|/\lambda) dx \leq 1 \right\} < \infty,$$

and the space  $H_{\varphi,L}(\mathbb{R}^n)$  is defined in the same way as  $H_L^{p(\cdot)}(\mathbb{R}^n)$  with  $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^n)}$  replaced by  $\|\cdot\|_{L^\varphi(\mathbb{R}^n)}$  (see [7]).



Observe that, if

$$(1.8) \quad \varphi(x, t) := t^{p(x)} \quad \text{for all } x \in \mathbb{R}^n \quad \text{and } t \in [0, \infty),$$

then  $L^\varphi(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n)$ . However, a general Musielak-Orlicz function  $\varphi$  satisfying all the assumptions in [34] (and hence [46]) may not have the form as in (1.8) (see [34]). On the other hand, it was proved in [47, Remark 2.23(iii)] that there exists a variable exponent function  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ , but  $t^{p(\cdot)}$  is not a uniformly Muckenhoupt weight, which was required in [46]. Thus, Musielak-Orlicz-Hardy spaces associated with operators in [46] and variable exponent Hardy spaces associated with operators in this article do not cover each other.

This article is organized as follows.

We first show Theorem 1.8 in Section 2 and then, as an application, we give out the proof of Theorem 1.11 in Section 3. Finally, in Section 4, applying Theorem 1.11, we prove Theorem 1.17.

We remark that, in the proof of Theorem 1.8, we borrow some ideas from [30, 31]. Precisely, to establish the atomic characterization of  $H_L^{p(\cdot)}(\mathbb{R}^n)$ , we need to use the Calderón reproducing formula associated with  $L$  (see (2.12) below) and the atomic decomposition of the variable tent space  $T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})$  established in [50, Theorem 2.16] (see also Lemma 2.1 below). Moreover, we show that the project operator  $\pi_{\Phi, L, M}$  is bounded from  $T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})$  to  $H_L^{p(\cdot)}(\mathbb{R}^n)$  by proving that, for any  $(p(\cdot), \infty)$ -atom  $a$  corresponding to the tent space  $T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})$ ,  $\pi_{\Phi, L, M}(a)$  is a  $(p(\cdot), q, M)_L$ -atom up to a positive constant multiple (see Proposition 2.5 below). We point out that Lemma 2.10 below obtained by Sawano [39, Lemma 4.1] plays a key role in the proofs of Proposition 2.5 and Theorem 1.8.

The strategy of the proof of Theorem 1.11 is presented in the following chains of inclusion relations:

$$(1.9) \quad \begin{aligned} \left[ H_{L, \text{at}, M}^{p(\cdot), q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right] &\subset \left[ H_{L, \max}^{p(\cdot), \mathcal{F}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right] \subset \left[ H_{L, \max}^{p(\cdot), \phi, a}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right] \\ &\subset \left[ H_{L, \text{at}, M}^{p(\cdot), \infty}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right] \subset \left[ H_{L, \text{at}, M}^{p(\cdot), q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right]. \end{aligned}$$

The second and the fourth inclusions are obviously. We prove the first inclusion in (1.9) via borrowing some ideas from the proof of [40, Theorem 1.4] and the third one by establishing a pointwise estimate for the non-tangential maximal function of any  $(p(\cdot), \infty, M)_L$ -atom.

The main step in the proof of Theorem 1.17 is to prove that, for all  $f \in L^2(\mathbb{R}^n)$ ,

$$(1.10) \quad \|f_{L, \nabla}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f_{L, +}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

via a modified technical based on the proof of [26, Theorem 2.1.4(b)]. Indeed, to obtain the inequality (1.10), for all  $f \in L^2(\mathbb{R}^n)$ , we first introduce a maximal function  $f_{L, \nabla}^{*, \epsilon, N}$  of  $f$ , where  $\epsilon, N \in (0, \infty)$ , which is a truncated version of the non-tangential maximal function  $f_{L, \nabla}^*$  (see (4.4) below). Then, under Assumption 1.15, we investigate the relation between  $f_{L, \nabla}^{*, \epsilon, N}$  and  $f_{L, +}^*$  in Lemma 4.4 below, which is further applied to prove the above inequality.

Here, we point out that the method used in the proof of (1.10) is different from that of the case  $p(\cdot) \equiv \text{constant} \in (0, 1]$ , which, as a special case, was essentially proved in [46, Theorem 1.9]. Indeed, in [46, Theorem 1.9], Yang et al. considered the Musielak-Orlicz Hardy spaces  $H_{\varphi, L}(\mathbb{R}^n)$  associated with the operator  $L$  satisfying the same assumptions as in the present article. Moreover, the approach used in the proof of [46, Theorem 1.9] strongly depends on the properties of uniformly Muckenhoupt weights, which are not possessed by  $t^{p(\cdot)}$  (see Remark 1.19).

At the end of this section, we make some conventions on notation. Let  $\mathbb{N} := \{1, 2, \dots\}$ ,  $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$  and  $\mathbb{R}_+^{n+1} := \mathbb{R}^n \times (0, \infty)$ . We denote by  $C$  a *positive constant* which is independent of the main parameters, but may vary from line to line. The *symbol*  $A \lesssim B$  means  $A \leq CB$ . If  $A \lesssim B$  and  $B \lesssim A$ , then we write  $A \sim B$ . We use  $C_{(\alpha, \dots)}$  to denote a positive constant depending on the indicated parameters  $\alpha, \dots$ . If  $E$  is a subset of  $\mathbb{R}^n$ , we denote by  $\chi_E$  its *characteristic function* and by  $E^c$  the set  $\mathbb{R}^n \setminus E$ . For  $a \in \mathbb{R}$ ,  $[a]$  denotes the largest integer  $m$  such that  $m \leq a$ . For all  $x \in \mathbb{R}^n$  and  $r \in (0, \infty)$ , denote by  $Q(x, r)$  the cube centered at  $x$  with side length  $r$ , whose sides are parallel to the axes of coordinates, and by  $B(x, r)$  the ball, namely,  $B(x, r) := \{y \in \mathbb{R}^n : |y - x| < r\}$ . For each cube  $Q \subset \mathbb{R}^n$  and  $a \in (0, \infty)$ , we use  $x_Q$  to denote the center of  $Q$  and  $\ell(Q)$  the side length of  $Q$ , and we also denote by  $aQ$  the cube concentric with  $Q$  having the side length  $a\ell(Q)$ .

## 2 Proof of Theorem 1.8

To prove Theorem 1.8, we first recall some notions about the variable exponent tent space introduced in [50].

Let  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ . For all measurable functions  $g$  on  $\mathbb{R}_+^{n+1}$  and  $x \in \mathbb{R}^n$ , define

$$\mathcal{T}(g)(x) := \left\{ \int_{\Gamma(x)} |g(y, t)|^2 \frac{dy dt}{t^{n+1}} \right\}^{1/2},$$

where  $\Gamma(x)$  is as in (1.3). Then the *variable exponent tent space*  $T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})$  is defined to be the set of all measurable functions  $g$  on  $\mathbb{R}_+^{n+1}$  such that  $\|g\|_{T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})} := \|\mathcal{T}(g)\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty$ . Recall that, for any  $q \in (0, \infty)$  being a constant exponent, the *tent space*  $T_2^q(\mathbb{R}_+^{n+1})$  was introduced in [11], which is defined to be the set of all measurable functions  $g$  on  $\mathbb{R}_+^{n+1}$  such that  $\|g\|_{T_2^q(\mathbb{R}_+^{n+1})} := \|\mathcal{T}(g)\|_{L^q(\mathbb{R}^n)} < \infty$ . Moreover, if  $g \in T_2^2(\mathbb{R}_+^{n+1})$ , then we easily know that

$$\|g\|_{T_2^2(\mathbb{R}_+^{n+1})} = \left\{ \int_{\mathbb{R}_+^{n+1}} |g(x, t)|^2 \frac{dx dt}{t} \right\}^{\frac{1}{2}}.$$

Let  $q \in (1, \infty)$  and  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ . Recall that a measurable function  $a$  on  $\mathbb{R}_+^{n+1}$  is called a  $(p(\cdot), q)$ -atom if  $a$  satisfies that  $\text{supp } a \subset \widehat{Q}$  for some cube  $Q \subset \mathbb{R}^n$  and

$$\|a\|_{T_2^q(\mathbb{R}_+^{n+1})} \leq |Q|^{1/q} \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1},$$

here and hereafter, for any cube  $Q \subset \mathbb{R}^n$ ,  $\widehat{Q}$  denotes the *tent* over  $Q$ , namely,

$$\widehat{Q} := \{(y, t) \in \mathbb{R}_+^{n+1} : B(y, t) \subset Q\}.$$

Furthermore, if  $a$  is a  $(p(\cdot), q)$ -atom for all  $q \in (1, \infty)$ , then  $a$  is called a  $(p(\cdot), \infty)$ -atom. We point out that the notion of  $(p(\cdot), \infty)$ -atoms was introduced in [50].

For any  $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ ,  $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$  and  $\{Q_j\}_{j \in \mathbb{N}}$  of cubes in  $\mathbb{R}^n$ , let

$$\mathcal{A}(\{\lambda_j\}_{j \in \mathbb{N}}, \{Q_j\}_{j \in \mathbb{N}}) := \left\| \left\{ \sum_{j \in \mathbb{N}} \left[ \frac{|\lambda_j| \chi_{Q_j}}{\|Q_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where  $\underline{p} := \min\{1, p_-\}$ .

The following atomic characterization of the space  $T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})$  was obtained in [48, Corollary 3.7].

**Lemma 2.1.** *Let  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Then  $f \in T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})$  if and only if there exist sequences  $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$  and  $\{a_j\}_{j \in \mathbb{N}}$  of  $(p(\cdot), \infty)$ -atoms such that, for almost every  $(x, t) \in \mathbb{R}_+^{n+1}$ ,*

$$(2.1) \quad f(x, t) = \sum_{j \in \mathbb{N}} \lambda_j a_j(x, t)$$

and

$$\int_{\mathbb{R}^n} \left\{ \sum_{j \in \mathbb{N}} \left[ \frac{\lambda_j \chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{\frac{p(x)}{\underline{p}}} dx < \infty,$$

where, for each  $j$ ,  $Q_j$  denotes the cube appearing in the support of  $a_j$ ; moreover, for all  $f \in T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})$ ,  $\|f\|_{T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})} \sim \mathcal{A}(\{\lambda_j\}_{j \in \mathbb{N}}, \{Q_j\}_{j \in \mathbb{N}})$  with the implicit equivalent positive constants independent of  $f$ .

In what follows, let  $T_{2,c}^{p(\cdot)}(\mathbb{R}_+^{n+1})$  and  $T_{2,c}^q(\mathbb{R}_+^{n+1})$  with  $q \in (0, \infty)$  be the sets of all functions in  $T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})$ , respectively,  $T_2^q(\mathbb{R}_+^{n+1})$  with compact supports.

**Remark 2.2.** Let  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ .

(i) It is known that  $T_{2,c}^{p(\cdot)}(\mathbb{R}_+^{n+1}) \subset T_{2,c}^2(\mathbb{R}_+^{n+1})$  as sets (see [48, Proposition 3.9]).

(ii) By [48, Corollary 3.4], we know that, for all  $f \in T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})$ , the decomposition (2.1) also holds true in  $T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})$ , which, in the case that  $p(\cdot) \equiv \text{constant} \in (0, \infty)$ , was proved by Jiang and Yang in [30, Proposition 3.1].

For a non-negative self-adjoint operator  $L$  on  $L^2(\mathbb{R}^n)$ , denoted by  $E_L$  the spectral measure associated with  $L$ . Then, for any bounded Borel measurable function  $F : [0, \infty) \rightarrow \mathbb{C}$ , the operator  $F(L) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$  is defined by the formula

$$F(L) := \int_0^\infty F(\lambda) dE_L(\lambda).$$

Let  $\phi_0 \in \mathcal{S}(\mathbb{R})$  be a given even function and  $\text{supp } \phi_0 \subset (-1, 1)$ . Assume that  $\Phi$  denotes the Fourier transform of  $\phi_0$ , namely, for all  $\xi \in \mathbb{R}^n$ ,  $\Phi(\xi) := \int_{\mathbb{R}^n} \phi_0(x) e^{-ix \cdot \xi} dx$ . For all  $f \in L^2(\mathbb{R}_+^{n+1})$  having compact support and  $x \in \mathbb{R}^n$ , define

$$\pi_{\Phi, L, M}(f)(x) := C_{(\Phi, M)} \int_0^\infty (t^2 L)^{M+1} \Phi(t\sqrt{L})(f(\cdot, t))(x) \frac{dt}{t},$$

where  $C_{(\Phi, M)}$  is the positive constant such that

$$(2.2) \quad 1 = C_{(\Phi, M)} \int_0^\infty t^{2(M+1)} \Phi(t) t^2 e^{-t^2} \frac{dt}{t}.$$

We then have the following lemma, which is a part of [28, Lemma 3.5].

**Lemma 2.3.** *Let  $\phi_0 \in \mathcal{S}(\mathbb{R})$  be an even function and  $\text{supp } \phi_0 \subset (-1, 1)$ . Assume that  $\Phi$  denotes the Fourier transform of  $\phi_0$ . Then, for any  $k \in \mathbb{Z}_+$ , the kernels  $\{K_{(t^2 L)^k \Phi(t\sqrt{L})}\}_{t>0}$  of the operators  $\{(t^2 L)^k \Phi(t\sqrt{L})\}_{t>0}$  satisfy that there exists a positive constant  $C$  such that, for all  $t \in (0, \infty)$  and  $x, y \in \mathbb{R}^n$ ,*

$$\text{supp } \left( K_{(t^2 L)^k \Phi(t\sqrt{L})} \right) \subset \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq t\}.$$

**Remark 2.4.** The operator  $\pi_{\Phi, L, M}$ , initially defined on  $T_{2, c}^2(\mathbb{R}_+^{n+1})$ , extends to a bounded linear operator from  $T_2^2(\mathbb{R}_+^{n+1})$  to  $L^2(\mathbb{R}^n)$  (see [31, Proposition 4.2(ii)]).

Moreover, we have the following conclusion.

**Proposition 2.5.** *Let  $L$  and  $p(\cdot)$  be as in Definition 1.5.*

(i) *Let  $M \in \mathbb{N}$  and  $a$  be a  $(p(\cdot), \infty)$ -atom. Then  $\pi_{\Phi, L, M}(a)$  is a  $(p(\cdot), \infty, M)_L$ -atom up to a positive constant multiple.*

(ii) *The operator  $\pi_{\Phi, L, M}$ , initially defined on  $T_{2, c}^{p(\cdot)}(\mathbb{R}_+^{n+1})$ , extends to a bounded linear operator from  $T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})$  to  $H_L^{p(\cdot)}(\mathbb{R}^n)$ .*

The proof of Proposition 2.5 depends on the following several lemmas. The following Fefferman-Stein vector-valued inequality of the Hardy-Littlewood maximal operator  $\mathcal{M}$  on the space  $L^{p(\cdot)}(\mathbb{R}^n)$  was obtained in [14, Corollary 2.1].

**Lemma 2.6.** *Let  $r \in (1, \infty)$  and  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ . If  $p_- \in (1, \infty)$  with  $p_-$  as in (1.1), then there exists a positive constant  $C$  such that, for all sequences  $\{f_j\}_{j=1}^\infty$  of measurable functions,*

$$\left\| \left\{ \sum_{j=1}^\infty [\mathcal{M}(f_j)]^r \right\}^{1/r} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \left\| \left( \sum_{j=1}^\infty |f_j|^r \right)^{1/r} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

**Remark 2.7.** Let  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$  and  $p_- \in (1, \infty)$ . Then there exists a positive constant  $C$  such that, for all  $f \in L^{p(\cdot)}(\mathbb{R}^n)$ ,  $\|\mathcal{M}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$  (see, for example, [18, Theorem 4.3.8]).

**Remark 2.8.** Let  $k \in \mathbb{N}$  and  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Then, by Lemma 2.6 and the fact that, for all cubes  $Q \subset \mathbb{R}^n$ ,  $r \in (0, p_-)$ ,  $\chi_{2^k Q} \leq 2^{kn/r} [\mathcal{M}(\chi_Q)]^{1/r}$ , we conclude that there exists a positive constant  $C$  such that, for any  $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$  and cubes  $\{Q_j\}_{j \in \mathbb{N}}$  of  $\mathbb{R}^n$ ,

$$\left\| \left\{ \sum_{j \in \mathbb{N}} \left[ \frac{|\lambda_j| \chi_{2^k Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{p_-} \right\}^{1/p_-} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C 2^{kn/r} \mathcal{A}(\{\lambda_j\}_{j \in \mathbb{N}}, \{Q_j\}_{j \in \mathbb{N}}).$$

The following lemma is just [50, Lemma 2.6].

**Lemma 2.9.** Let  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Then there exists a positive constant  $C$  such that, for all cubes  $Q_1$  and  $Q_2$  of  $\mathbb{R}^n$  with  $Q_1 \subset Q_2$ ,

$$C^{-1} \left( \frac{|Q_1|}{|Q_2|} \right)^{1/p_-} \leq \frac{\|\chi_{Q_1}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{Q_2}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C \left( \frac{|Q_1|}{|Q_2|} \right)^{1/p_+},$$

where  $p_-$  and  $p_+$  are as in (1.1).

We also need the following useful lemma, which is just [39, Lemma 4.1] and plays a key role in the present article.

**Lemma 2.10.** Let  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$  and  $q \in [1, \infty) \cap (p_+, \infty)$ , where  $p_+$  is as in (1.1). Then there exists a positive constant  $C$  such that, for all sequences  $\{Q_j\}_{j \in \mathbb{N}}$  of cubes,  $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$  and functions  $\{a_j\}_{j \in \mathbb{N}}$  satisfying that, for each  $j \in \mathbb{N}$ ,  $\text{supp } a_j \subset Q_j$  and  $\|a_j\|_{L^q(\mathbb{R}^n)} \leq |Q_j|^{1/q}$ ,

$$\left\| \left( \sum_{j=1}^{\infty} |\lambda_j a_j|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \left\| \left( \sum_{j=1}^{\infty} |\lambda_j \chi_{Q_j}|^2 \right)^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

We are now ready to prove Proposition 2.5.

*Proof of Proposition 2.5.* We first prove (i). Let  $a$  be a  $(p(\cdot), \infty)$ -atom associated with some cube  $Q \subset \mathbb{R}^n$ . Let

$$b := C_{(\Phi, M)} \int_0^\infty t^{2(M+1)} L\Phi(t\sqrt{L})(a(\cdot, t)) \frac{dt}{t},$$

where  $C_{(\Phi, M)}$  is as in (2.2). Then  $\pi_{\Phi, L, M}(a) = L^M(b)$ . By Lemma 2.3 and the fact that  $\text{supp } a \subset Q$ , we easily know that  $\text{supp } L^k b \subset \sqrt{n}Q$  for each  $k \in \{0, 1, \dots, M\}$ . On the other hand, by [11, Lemma 2] and the Hölder inequality, we find that, for any  $q \in (1, \infty)$  and  $h \in L^2(Q) \cap L^{q'}(Q)$  with  $q' := \frac{q}{q-1}$ ,

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} ([\ell(Q)]^2 L)^k b(x) h(x) dx \right| \\ & \lesssim [\ell(Q)]^{2M} \int_{\mathbb{R}^n} \int_0^{\ell(Q)} \left| a(y, t) (t^2 L)^{k+1} \Phi(t\sqrt{L})(h)(y) \right| \frac{dt}{t} dy \end{aligned}$$

$$\begin{aligned}
&\lesssim [\ell(Q)]^{2M} \int_{\mathbb{R}^n} \left\{ \int_{\Gamma(x)} |a(y, t)| \left| (t^2 L)^{k+1} \Phi(t\sqrt{L}) h(y) \right| \frac{dydt}{t^{n+1}} \right\} dx \\
&\lesssim [\ell(Q)]^{2M} \|a\|_{T_2^q(\mathbb{R}_+^{n+1})} \left\| \tilde{S}_L^k(h) \right\|_{L^{q'}(\mathbb{R}^n)} \lesssim \frac{[\ell(Q)]^{2M+n/q}}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|h\|_{L^{q'}(\mathbb{R}^n)},
\end{aligned}$$

where

$$\tilde{S}_L^k(h)(x) := \left\{ \int_{\Gamma(x)} \left| (t^2 L)^{k+1} \Phi(t\sqrt{L}) h(y) \right|^2 \frac{dydt}{t^{n+1}} \right\}^{1/2}$$

with  $\Gamma(x)$  as in (1.3), which is bounded on  $L^r(\mathbb{R}^n)$  with  $r \in (1, \infty)$  (see, for example, [7, Lemma 5.3]). Therefore,  $\pi_{\Phi, L, M}(a)$  is a  $(p(\cdot), \infty, M)_L$ -atom up to a positive constant multiple and hence the proof of (i) is completed.

Next, we show (ii). Let  $f \in T_{2,c}^{p(\cdot)}(\mathbb{R}_+^{n+1})$ . Then, by Remark 2.2(i), we know that  $f \in T_{2,c}^2(\mathbb{R}_+^{n+1})$  and hence, due to Remark 2.4,  $\pi_{\Phi, L, M}$  is well defined on  $T_{2,c}^{p(\cdot)}(\mathbb{R}_+^{n+1})$ . From this, combined with Lemma 2.1 and Remark 2.2(ii), we deduce that  $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$  in both  $T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})$  and  $T_2^2(\mathbb{R}_+^{n+1})$ , where  $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$  and  $\{a_j\}_{j \in \mathbb{N}}$  are  $(p(\cdot), \infty)$  atoms associated cubes  $\{Q_j\}_{j \in \mathbb{N}}$  of  $\mathbb{R}^n$  satisfying

$$(2.3) \quad \mathcal{A}(\{\lambda_j\}_{j \in \mathbb{N}}, \{Q_j\}_{j \in \mathbb{N}}) \lesssim \|f\|_{T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})};$$

moreover

$$\pi_{\Phi, L, M}(f) = \sum_{j \in \mathbb{N}} \lambda_j \pi_{\Phi, L, M}(a_j) =: \sum_{j \in \mathbb{N}} \lambda_j \alpha_j \quad \text{in } L^2(\mathbb{R}^n).$$

Obviously, for any  $j \in \mathbb{N}$ ,  $\alpha_j$  is a  $(p(\cdot), \infty, M)_L$ -atom up to a positive constant multiple by (i). Since  $S_L$  is bounded on  $L^2(\mathbb{R}^n)$  (see (1.4)), it follows that, for almost every  $x \in \mathbb{R}^n$ ,

$$S_L(\pi_{\Phi, L, M}(f))(x) \leq \sum_{j \in \mathbb{N}} |\lambda_j| S_L(\alpha_j)(x).$$

Thus, we have

$$\begin{aligned}
&\|S_L(\pi_{\Phi, L, M}(f))\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leq \left\| \sum_{j \in \mathbb{N}} |\lambda_j| S_L(\alpha_j) \chi_{4\sqrt{n}Q_j} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \left\| \sum_{j \in \mathbb{N}} |\lambda_j| S_L(\alpha_j) \chi_{(4\sqrt{n}Q_j)^c} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} =: \text{I} + \text{II}.
\end{aligned}$$

Observe that, by (1.4), we find that, for any  $q \in (1, \infty)$  and  $j \in \mathbb{N}$ ,

$$\|S_L(\alpha_j)\|_{L^q(\mathbb{R}^n)} \lesssim \|\alpha_j\|_{L^q(\mathbb{R}^n)} \lesssim \frac{|4\sqrt{n}Q_j|^{1/q}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}.$$

By this, Lemma 2.10, Remark 2.8 and (2.3), we conclude that

$$\text{I} \lesssim \left\| \left\{ \sum_{j \in \mathbb{N}} \left[ \frac{|\lambda_j| \chi_{4\sqrt{n}Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{p_-} \right\}^{1/p_-} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \mathcal{A}(\{\lambda_j\}_{j \in \mathbb{N}}, \{Q_j\}_{j \in \mathbb{N}}) \lesssim \|f\|_{T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})}.$$

To estimate  $\Pi$ , we first claim that, for all  $\delta \in (n[1/p_- - 1], 2M)$ ,  $j \in \mathbb{N}$  and  $x \in (4\sqrt{n}Q_j)^\complement$ ,

$$(2.4) \quad S_L(\alpha_j)(x) \lesssim \frac{[\ell(Q_j)]^{n+\delta}}{|x - x_{Q_j}|^{n+\delta}} \frac{1}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}.$$

Indeed, for all  $j \in \mathbb{N}$  and  $x \in (4\sqrt{n}Q_j)^\complement$ ,

$$(2.5) \quad [S_L(\alpha_j)(x)]^2 = \int_0^{\ell(Q_j)} \int_{|y-x|<t} \left| t^2 L e^{-t^2 L}(\alpha_j)(y) \right|^2 \frac{dy dt}{t^{n+1}} + \int_{\ell(Q_j)}^\infty \int_{|y-x|<t} \dots \\ =: \Pi_1(x) + \Pi_2(x).$$

Notice that  $t^2 L e^{-t^2 L} = (-r \frac{de^{-rL}}{dr})_{r=t^2}$ . It follows from (1.2) that

$$(2.6) \quad \begin{aligned} \Pi_1(x) &= \int_0^{\ell(Q_j)} \int_{|y-x|<t} \left| \left( r \frac{de^{-rL}}{dr} \right)_{r=t^2} (\alpha_j)(y) \right|^2 \frac{dy dt}{t^{n+1}} \\ &\lesssim \int_0^{\ell(Q_j)} \int_{|y-x|<t} \left[ \int_{Q_j} \frac{1}{t^n} e^{-\frac{|z-y|^2}{t^2}} |\alpha_j(z)| dz \right]^2 \frac{dy dt}{t^{n+1}} \\ &\lesssim \int_0^{\ell(Q_j)} \int_{|y-x|<t} \left[ \int_{Q_j} \frac{t^\delta}{(t+|z-y|)^{n+\delta}} |\alpha_j(z)| dz \right]^2 \frac{dy dt}{t^{n+1}}. \end{aligned}$$

Since, for all  $x \in (4\sqrt{n}Q_j)^\complement$ ,  $t \in (0, \infty)$ ,  $|y-x| < t$  and  $z \in Q_j$ , we have

$$(2.7) \quad t + |z-y| \geq |x-z| \geq |x-x_{Q_j}| - |x_{Q_j}-z| \geq |x-x_{Q_j}|/2.$$

By this, (2.6) and the Hölder inequality, we further find that

$$(2.8) \quad \begin{aligned} \Pi_1(x) &\lesssim \frac{|Q_j|^{2(1-1/q)}}{|x-x_{Q_j}|^{2(n+\delta)}} \|\alpha_j\|_{L^q(\mathbb{R}^n)}^2 \int_0^{\ell(Q_j)} t^{2\delta} \frac{dt}{t} \\ &\lesssim \frac{[\ell(Q_j)]^{2(n+\delta)}}{|x-x_{Q_j}|^{2(n+\delta)}} \frac{1}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^2}. \end{aligned}$$

On the other hand, by the proof of (i), we know that, for each  $j \in \mathbb{N}$ , there exists  $b_j \in \mathcal{D}(L^M)$  such that  $\alpha_j = L^M(b_j)$  and

$$\|b_j\|_{L^q(\mathbb{R}^n)} \lesssim [\ell(Q_j)]^{2M} |Q_j|^{1/q} \|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}.$$

From this, (1.2), (2.7) and the fact that

$$t^2 L^{M+1} e^{-t^2 L} = (-1)^{M+1} t^{-2M} \left( r^{M+1} \frac{d^{M+1} e^{-rL}}{dr^{M+1}} \right)_{r=t^2},$$

we deduce that

$$(2.9) \quad \Pi_2(x) \sim \int_{\ell(Q_j)}^\infty \int_{|y-x|<t} \left| \left( r^{M+1} \frac{d^{M+1} e^{-rL}}{dr^{M+1}} \right)_{r=t^2} (b_j)(y) \right|^2 \frac{dy dt}{t^{n+4M+1}}$$



$$\begin{aligned}
&\lesssim \int_{\ell(Q_j)}^\infty \int_{|y-x|<t} \left[ \int_{Q_j} \frac{1}{t^n} e^{-\frac{|z-y|^2}{t^2}} |b_j(z)| dz \right]^2 \frac{dydt}{t^{n+4M+1}} \\
&\lesssim \frac{|Q_j|^{2(1-1/q)}}{|x-x_{Q_j}|^{2(n+\delta)}} \|b_j\|_{L^q(\mathbb{R}^n)}^2 \int_{\ell(Q_j)}^\infty t^{2\delta} \frac{dt}{t^{4M+1}} \\
&\lesssim \frac{[\ell(Q_j)]^{2(n+\delta)}}{|x-x_{Q_j}|^{2(n+\delta)}} \frac{1}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^2}.
\end{aligned}$$

Combining (2.5), (2.8) and (2.9), we conclude that (2.4) holds true, which completes the proof of the above claim.

Now, let  $r \in (0, p_-)$  be such that  $\delta \in (n[1/r - 1], 2M)$ . Then, from the above claim, Remarks 1.1 and 2.8, and (2.3), we deduce that

$$\begin{aligned}
\Pi &\lesssim \left\| \sum_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} |\lambda_j| S_L(\alpha_j) \chi_{2^{k+2}\sqrt{n}Q_j \setminus (2^{k+1}\sqrt{n}Q_j)} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \left\{ \sum_{k \in \mathbb{N}} 2^{-k(n+\delta)p_-} \left\| \sum_{j \in \mathbb{N}} \left[ \frac{|\lambda_j| \chi_{2^{k+2}\sqrt{n}Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{p_-} \right\|_{L^{p(\cdot)/p_-}(\mathbb{R}^n)} \right\}^{1/p_-} \\
&\lesssim \left\{ \sum_{k \in \mathbb{N}} 2^{-k(n+\delta)p_-} 2^{knp_-/r} \left\| \left\{ \sum_{j \in \mathbb{N}} \left[ \frac{|\lambda_j| \chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{p_-} \right\}^{1/p_-} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_-} \right\}^{1/p_-} \\
&\lesssim \|f\|_{T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})} \left\{ \sum_{k \in \mathbb{N}} 2^{-k(n+\delta-n/r)} \right\}^{1/p_-} \sim \|f\|_{T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})}.
\end{aligned}$$

This, together with the estimate of I, implies that

$$(2.10) \quad \|S_L(\pi_{\Phi, L, M}(f))\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})}$$

and hence (ii) holds true, which completes the proof of Proposition 2.5.  $\square$

We now prove Theorem 1.8 by using Proposition 2.5.

*Proof of Theorem 1.8.* We first show

$$(2.11) \quad H_{L, \text{at}, M}^{p(\cdot), q}(\mathbb{R}^n) \subset H_L^{p(\cdot)}(\mathbb{R}^n).$$

Let  $f \in H_{L, \text{at}, M}^{p(\cdot), q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ . Then, by Definition 1.6, we know that  $f$  has a representation  $f = \sum_{j \in \mathbb{N}} \lambda_j \alpha_j$  in  $L^2(\mathbb{R}^n)$ , where  $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$  and  $\{\alpha_j\}_{j \in \mathbb{N}}$  are  $(p(\cdot), q, M)_L$ -atoms such that  $\mathcal{B}(\{\lambda_j \alpha_j\}_{j \in \mathbb{N}}) \lesssim \|f\|_{H_{L, \text{at}, M}^{p(\cdot), q}(\mathbb{R}^n)}$ . By an argument similar to that used in the proof of Proposition 2.5(ii), we conclude that  $\|S_L(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H_{L, \text{at}, M}^{p(\cdot), q}(\mathbb{R}^n)}$ , which implies that

$$\left[ H_{L, \text{at}, M}^{p(\cdot), q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right] \subset H_L^{p(\cdot)}(\mathbb{R}^n)$$

and hence (2.11) holds true by Remark 1.14.

Conversely, we need to show

$$(2.12) \quad H_L^{p(\cdot)}(\mathbb{R}^n) \subset H_{L,\text{at},M}^{p(\cdot),q}(\mathbb{R}^n).$$

Let  $f \in H_L^{p(\cdot)}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ . Then, by the functional calculi for  $L$ , we know that

$$f = C_{(\Phi,M)} \int_0^\infty (t^2 L)^{M+1} \Phi(t\sqrt{L})(t^2 L e^{-t^2 L} f) \frac{dt}{t} = \pi_{(\Phi,L,M)}(t^2 L e^{-t^2 L} f) \quad \text{in } L^2(\mathbb{R}^n),$$

where  $C_{(\Phi,M)}$  is as in (2.2). Since  $S_L(f) \in L^{p(\cdot)}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ , it follows that  $t^2 L e^{-t^2 L}(f) \in T_2^{p(\cdot)}(\mathbb{R}_+^{n+1}) \cap T_2^2(\mathbb{R}_+^{n+1})$ . Thus, from Lemma 2.1 and Proposition 2.5(ii), we deduce that there exist sequences  $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$  and  $\{a_j\}_{j \in \mathbb{N}}$  of  $(p(\cdot), \infty)$ -atoms such that

$$f = \pi_{\Phi,L,M}(t^2 L e^{-t^2 L} f) = \sum_{j \in \mathbb{N}} \lambda_j \pi_{\Phi,L,M}(a_j) =: \sum_{j \in \mathbb{N}} \lambda_j \alpha_j \quad \text{in both } L^2(\mathbb{R}^n) \text{ and } H_L^{p(\cdot)}(\mathbb{R}^n),$$

and

$$\mathcal{B}(\{\lambda_j \alpha_j\}_{j \in \mathbb{N}}) \lesssim \|t^2 L e^{-t^2 L} f\|_{T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})} \sim \|f\|_{H_L^{p(\cdot)}(\mathbb{R}^n)}.$$

On the other hand, by Proposition 2.5(i), we know that, for  $j \in \mathbb{N}$ ,  $\alpha_j$  is a  $(p(\cdot), \infty, M)_L$ -atom up to a positive constant multiple. Therefore,  $f \in H_{L,\text{at},M}^{p(\cdot),\infty}(\mathbb{R}^n) \subset H_{L,\text{at},M}^{p(\cdot),q}(\mathbb{R}^n)$ , which implies that (2.12) holds true and hence completes the proof of Theorem 1.8.  $\square$

### 3 Proof of Theorem 1.11

In this section, we prove Theorem 1.11. We first recall the following notion.

For a given Borel measurable function  $F$  on  $\mathbb{R}_+^{n+1}$ , the *non-tangential maximal function* of  $F$  with aperture  $\alpha \in (0, \infty)$  is defined by setting, for all  $x \in \mathbb{R}^n$ ,

$$(3.1) \quad M_\alpha(F)(x) := \sup_{t \in (0, \infty), |y-x| < \alpha t} |F(y, t)|.$$

**Lemma 3.1.** *Let  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$  and  $\alpha_1, \alpha_2 \in (0, \infty)$ . If  $\lambda \in (n/p_-, \infty)$ , then there exists a positive constant  $C$  such that, for any Borel measurable function  $F$  on  $\mathbb{R}_+^{n+1}$ ,*

$$(3.2) \quad \|M_{\alpha_1}(F)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \left(1 + \frac{\alpha_1}{\alpha_2}\right)^\lambda \|M_{\alpha_2}(F)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

*Proof.* For any  $\alpha \in (0, \infty)$  and  $\lambda \in (n/p_-, \infty)$ , let

$$(3.3) \quad N_\lambda^\alpha(F)(x) := \sup_{t \in (0, \infty), y \in \mathbb{R}^n} |F(y, t)| \left(1 + \frac{|x-y|}{\alpha t}\right)^{-\lambda}.$$

Then it is easy to see that  $M_\alpha(F)(x) \lesssim N_\lambda^\alpha(F)(x)$  for all  $x \in \mathbb{R}^n$ .

Therefore, to prove (3.2), it suffices to show that, for any  $\alpha_1, \alpha_2 \in (0, \infty)$ ,

$$(3.4) \quad \|N_\lambda^{\alpha_1}(F)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \left(1 + \frac{\alpha_1}{\alpha_2}\right)^\lambda \|M_{\alpha_2}(F)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

To prove this, we first notice that, for any  $t \in (0, \infty)$ ,  $x, y \in \mathbb{R}^n$  and all  $z \in B(x - y, \alpha_2 t)$ ,

$$|F(x - y, t)| \leq M_{\alpha_2}(F)(z).$$

Then, since  $B(x - y, \alpha_2 t) \subset B(x, |y| + \alpha_2 t)$ , it follows that

$$\begin{aligned} |F(x - y, t)|^{\frac{n}{\lambda}} &\leq \frac{1}{|B(x - y, \alpha_2 t)|} \int_{B(x, |y| + \alpha_2 t)} |M_{\alpha_2}(F)(z)|^{\frac{n}{\lambda}} dz \\ &\leq \frac{|B(x, |y| + \alpha_2 t)|}{|B(x - y, \alpha_2 t)|} \mathcal{M}([M_{\alpha_2}(F)]^{n/\lambda})(x) \\ &\lesssim \left(1 + \frac{\alpha_1}{\alpha_2}\right)^n \left(1 + \frac{|y|}{\alpha_1 t}\right)^n \mathcal{M}([M_{\alpha_2}(F)]^{n/\lambda})(x). \end{aligned}$$

Thus, we conclude that, for all  $t \in (0, \infty)$  and  $x, y \in \mathbb{R}^n$ ,

$$|F(x - y, t)| \left(1 + \frac{|y|}{\alpha_1 t}\right)^{-\lambda} \lesssim \left(1 + \frac{\alpha_1}{\alpha_2}\right)^\lambda \left\{ \mathcal{M}([M_{\alpha_2}(F)]^{n/\lambda})(x) \right\}^{\frac{\lambda}{n}},$$

which further implies that

$$N_\lambda^{\alpha_1}(F)(x) \lesssim \left(1 + \frac{\alpha_1}{\alpha_2}\right)^\lambda \left\{ \mathcal{M}([M_{\alpha_2}(F)]^{n/\lambda})(x) \right\}^{\frac{\lambda}{n}}.$$

From this, Remark 2.7 and the fact that  $\lambda \in (n/p_-, \infty)$ , we deduce that (3.4) holds true, which completes the proof of Lemma 3.1.  $\square$

**Remark 3.2.** When  $p(\cdot) \equiv \text{constant} \in (0, \infty)$ , Lemma 3.1 was established by Calderón and Torchinsky in [9, Theorem 2.3].

By Lemma 3.1, we immediate obtain the following conclusion, the details being omitted.

**Corollary 3.3.** *Let  $L$  be as in Theorem 1.11,  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ ,  $\alpha_1, \alpha_2 \in (0, \infty)$  and  $\varphi \in \mathcal{S}(\mathbb{R})$  be an even function with  $\varphi(0) = 1$ . If  $\lambda \in (n/p_-, \infty)$ , then there exists a positive constant  $C$  such that, for all  $f \in L^2(\mathbb{R}^n)$ ,*

$$(3.5) \quad \|\varphi_{L, \nabla, \alpha_1}^*(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \left(1 + \frac{\alpha_1}{\alpha_2}\right)^\lambda \|\varphi_{L, \nabla, \alpha_2}^*(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

We also have the following technical lemma.

**Lemma 3.4.** *Let  $L$  be as in Theorem 1.11,  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ ,  $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R})$  be even functions with  $\psi_1(0) = 1 = \psi_2(0)$  and  $\alpha_1, \alpha_2 \in (0, \infty)$ . Then there exists a positive constant  $C \in (0, \infty)$ , depending on  $\psi_1, \psi_2, \alpha_1$  and  $\alpha_2$ , such that, for all  $f \in L^2(\mathbb{R}^n)$ ,*

$$(3.6) \quad \|(\psi_1)_{L, \nabla, \alpha_1}^*(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|(\psi_2)_{L, \nabla, \alpha_2}^*(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

*Proof.* Let  $\psi := \psi_1 - \psi_2$ . Then, by Remark 1.1, we have

$$\|(\psi_1)_{L,\nabla,\alpha_1}^*(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|\psi_{L,\nabla,\alpha_1}^*(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \|(\psi_2)_{L,\nabla,\alpha_1}^*(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Thus, to prove (3.6), by Corollary 3.3, it suffices to show that

$$(3.7) \quad \|\psi_{L,\nabla,\alpha_1}^*(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|(\psi_2)_{L,\nabla,\alpha_2}^*(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Moreover, due to (3.5), we may assume that  $\alpha_1 = 1 = \alpha_2$ . Then, by [40, (3.3) and (3.4)], we find that, for all  $\lambda \in (n/p_-, \infty)$  and  $x \in \mathbb{R}^n$ ,

$$\psi_{L,\nabla,1}^*(f)(x) \lesssim N_\lambda^1 \left( \psi_2(t\sqrt{L}) \right) (f)(x),$$

which, together with (3.4), implies that (3.7) holds true. This finishes the proof of Lemma 3.4.  $\square$

We now show Theorem 1.11.

*Proof of Theorem 1.11.* We first prove that, for any  $q \in (1, \infty]$  and  $M \in (\frac{n}{2}[\frac{1}{p_-} - 1], \infty) \cap \mathbb{N}$ ,

$$(3.8) \quad \left[ H_{L,\text{at},M}^{p(\cdot),q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right] \subset \left[ H_{L,\max}^{p(\cdot),\mathcal{F}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right].$$

Let  $f \in H_{L,\text{at},M}^{p(\cdot),q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ . Then  $f$  has a representation:  $f = \sum_{j \in \mathbb{N}} \lambda_j \alpha_j$  in  $L^2(\mathbb{R}^n)$ , where  $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$  and  $\{\alpha_j\}_{j \in \mathbb{N}}$  is a sequence of  $(p(\cdot), q, M)_L$ -atoms associated with cubes  $\{Q_j\}_{j \in \mathbb{N}}$  of  $\mathbb{R}^n$  such that  $\mathcal{B}(\{\lambda_j \alpha_j\}_{j \in \mathbb{N}}) \lesssim \|f\|_{H_L^{p(\cdot)}(\mathbb{R}^n)}$ .

For any  $\phi \in \mathcal{F}(\mathbb{R})$  and  $x \in \mathbb{R}^n$ , let  $\tilde{\psi}(x) := [\phi(0)]^{-1} \phi(x) - e^{-x^2}$ . Then, by an argument similar to that used in the proof of [40, (3.4)] (see also [46, p.18]), we conclude that, for any  $\lambda \in (0, \infty)$ , there exists a positive constant  $C$ , depending on  $n, \Psi$  and  $\lambda$ , such that, for all  $\phi \in \mathcal{F}(\mathbb{R})$ ,

$$\sup_{|w| < t} \int_{\mathbb{R}_+^{n+1}} \left| K_{\tilde{\psi}(t\sqrt{L})\Psi(s\sqrt{L})}(x-w, z) \right| \left[ 1 + \frac{|x-z|}{s} \right]^\lambda \frac{dz ds}{s} \leq C,$$

where  $\Psi$  is as in Lemma 2.3. From this estimate and some arguments similar to those used in the proofs of (3.7) and [40, (3.3) and (3.4)], we deduce that

$$\left\| \sup_{\phi \in \mathcal{F}(\mathbb{R})} \tilde{\psi}_{L,\nabla,1}^*(f) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f_{L,\nabla}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where  $f_{L,\nabla}^*$  is as in Definition 1.9. Since

$$\mathcal{G}_{L,\nabla}^*(f) \lesssim \sup_{\phi \in \mathcal{F}(\mathbb{R})} \tilde{\psi}_{L,\nabla,1}^*(f) + f_{L,\nabla}^*,$$

it follows that, to prove  $\|\mathcal{G}_{L,\nabla}^*(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H_L^{p(\cdot)}(\mathbb{R}^n)}$ , we only need to show that

$$(3.9) \quad \|f_{L,\nabla}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H_L^{p(\cdot)}(\mathbb{R}^n)}.$$

To prove this, we claim that, for any  $(p(\cdot), q, M)_L$ -atom  $\alpha$  associated with some cube  $Q := Q(x_Q, \ell(Q)) \subset \mathbb{R}^n$  for some  $x_Q \in \mathbb{R}^n$  and  $\ell(Q) \in (0, \infty)$  and  $x \in (4\sqrt{n}Q)^\complement$ ,

$$(3.10) \quad \alpha_{L, \nabla}^*(x) \lesssim \frac{[\ell(Q)]^{n+\delta}}{|x - x_Q|^{n+\delta}} \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}},$$

where  $\delta \in (n[1/p_- - 1], 2M)$ . If this claim holds true, then, observing that

$$\|f_{L, \nabla}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \left\| \sum_{j \in \mathbb{N}} |\lambda_j| (\alpha_j)_{L, \nabla}^* \chi_{4\sqrt{n}Q_j} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \left\| \sum_{j \in \mathbb{N}} |\lambda_j| (\alpha_j)_{L, \nabla}^* \chi_{(4\sqrt{n}Q_j)^\complement} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

by Remark 1.10 and some argument similar to that used in the proof of (2.10), we conclude that (3.9) holds true.

Therefore, it remains to prove the above claim. For any given  $x \in (4\sqrt{n}Q)^\complement$ , let

$$\alpha_{L, \nabla}^{*,1}(x) := \sup_{t \in (0, \ell(Q)), |y-x| < t} \left| e^{-t^2 L}(\alpha)(y) \right|$$

and

$$\alpha_{L, \nabla}^{*,2}(x) := \sup_{t \in [\ell(Q), \infty), |y-x| < t} \left| e^{-t^2 L}(\alpha)(y) \right|.$$

Notice that, for all  $t \in (0, \infty)$ ,  $z \in Q$  and  $y \in \mathbb{R}^n$  with  $|y - x| < t$ , we have

$$(3.11) \quad t + |y - z| > |x - z| \geq |x - x_Q| - |z - x_Q| \geq |x - x_Q|/2.$$

Then, by Assumption 1.3 and the Hölder inequality, we find that

$$(3.12) \quad \begin{aligned} \alpha_{L, \nabla}^{*,1}(x) &\lesssim \sup_{t \in (0, \ell(Q)), |y-x| < t} \int_{\mathbb{R}^n} \frac{1}{t^n} e^{-\frac{|y-z|^2}{ct^2}} |\alpha(z)| dz \\ &\lesssim \sup_{t \in (0, \ell(Q)), |y-x| < t} \int_Q \frac{t^\delta}{(t + |y - z|)^{n+\delta}} |\alpha(z)| dz \\ &\lesssim \frac{[\ell(Q)]^\delta}{|x - x_Q|^{n+\delta}} |Q|^{1-1/q} \|\alpha\|_{L^q(\mathbb{R}^n)} \lesssim \frac{[\ell(Q)]^{n+\delta}}{|x - x_Q|^{n+\delta}} \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}. \end{aligned}$$

On the other hand, letting  $\alpha := L^M b$  be as in Definition 1.6, from (1.2), (3.11) and the fact that  $M > \delta/2$ , we deduce that

$$\begin{aligned} \alpha_{L, \nabla}^{*,2}(x) &= \sup_{t \in [\ell(Q), \infty), |y-x| < t} \left| e^{-t^2 L} L^M b(y) \right| \\ &= \sup_{t \in [\ell(Q), \infty), |y-x| < t} t^{-2M} \left| (t^2 L)^M e^{-t^2 L}(b)(y) \right| \\ &\lesssim \sup_{t \in [\ell(Q), \infty), |y-x| < t} t^{-2M} \int_Q \frac{t^\delta}{(t + |z - y|)^{n+\delta}} |b(z)| dz \\ &\lesssim \sup_{t \in [\ell(Q), \infty), |y-x| < t} \frac{t^{\delta-2M}}{|x - x_Q|^{n+\delta}} |Q|^{1-1/q} \|b\|_{L^q(\mathbb{R}^n)} \end{aligned}$$

$$\lesssim \frac{[\ell(Q)]^{n+\delta}}{|x - x_Q|^{n+\delta}} \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}.$$

By this and (3.12), we conclude that, for all  $x \in (4\sqrt{n}Q)^\complement$ ,

$$\alpha_{L,\nabla}^*(x) \leq \alpha_{L,\nabla}^{*,1}(x) + \alpha_{L,\nabla}^{*,2}(x) \lesssim \frac{[\ell(Q)]^{n+\delta}}{|x - x_Q|^{n+\delta}} \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}},$$

namely, (3.10) holds true. This finishes the proof of (3.8).

Next, we show that

$$(3.13) \quad \left[ H_{L,\max}^{p(\cdot),\phi,a}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right] \subset \left[ H_{L,\text{at},M}^{p(\cdot),\infty}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right].$$

To this end, by Lemma 3.4, it suffices to prove that, if  $f \in H_{L,\max}^{p(\cdot)}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ , then  $f \in H_{L,\text{at},M}^{p(\cdot),\infty}(\mathbb{R}^n)$  and

$$(3.14) \quad \|f\|_{H_{L,\text{at},M}^{p(\cdot),\infty}(\mathbb{R}^n)} \lesssim \|f\|_{H_{L,\max}^{p(\cdot)}(\mathbb{R}^n)}.$$

Let  $\Phi$  be a function as in Lemma 2.3 and, for all  $x \in \mathbb{R}$ ,  $\Psi(x) := x^{2M}\Phi(x)$ . Then, by the functional calculi, we know that there exists a constant  $C_{(\Psi)}$  such that

$$f = C_{(\Psi)} \int_0^\infty \Psi(t\sqrt{L}) t^2 L e^{-t^2 L} (f) \frac{dt}{t} \quad \text{in } L^2(\mathbb{R}^n).$$

Define a function  $\eta$  by setting, when  $x \in \mathbb{R} \setminus \{0\}$ ,

$$\eta(x) := C_{(\Psi)} \int_1^\infty t^2 x^2 \Psi(tx) e^{-t^2 x^2} \frac{dt}{t}$$

and  $\eta(0) = 1$ . Then  $\eta \in \mathcal{S}(\mathbb{R})$  is an even function and, for any  $a, b \in \mathbb{R}$ ,

$$\eta(ax) - \eta(bx) = C_{(\Psi)} \int_a^b t^2 x^2 \Psi(tx) e^{-t^2 x^2} \frac{dt}{t},$$

which implies that

$$C_{(\Psi)} \int_a^b \Psi(t\sqrt{L}) t^2 L e^{-t^2 L} (f) \frac{dt}{t} = \eta(a\sqrt{L})(f) - \eta(b\sqrt{L})(f).$$

Let, for all  $x \in \mathbb{R}^n$ ,

$$\mathcal{N}_L^*(f)(x) := \sup_{t \in (0, \infty), |y-x| < 5\sqrt{n}t} \left( \left| t^2 L e^{-t^2 L} f(y) \right| + \left| \eta(t\sqrt{L}) f(y) \right| \right).$$

Then, by Lemma 3.4, we know that

$$(3.15) \quad \|\mathcal{N}_L^*(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f_{L,\nabla}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|f\|_{H_{L,\max}^{p(\cdot)}(\mathbb{R}^n)}.$$

Now, following [40, p.476], for  $i \in \mathbb{Z}$ , let  $O_i := \{x \in \mathbb{R}^n : \mathcal{N}_L^*(f)(x) \geq 2^i\}$ . Denote by  $\{Q_{i,j}\}_{j \in \mathbb{N}}$  the Whitney decomposition of  $O_i$ . For each  $i \in \mathbb{Z}$  and  $j \in \mathbb{N}$ , let

$$\widehat{O}_i := \{(x, t) \in \mathbb{R}^n : B(x, 4\sqrt{n}t) \subset O_i\}$$

and

$$\widetilde{Q}_{i,j} := \{(y, t) \in \mathbb{R}_+^{n+1} : y + 3te_0 \in Q_{i,j}\},$$

here and hereafter,  $e_0 := \overbrace{(1, \dots, 1)}^{n \text{ times}} \in \mathbb{R}^n$ . Then it is easy to prove that  $\widehat{O}_i \subset \bigcup_{j \in \mathbb{N}} \widetilde{Q}_{i,j}$  (see [40, p.476] for more details). Observe that, for each fixed  $i \in \mathbb{Z}$ , when  $j \neq k$ ,  $Q_{i,j} \cap Q_{i,k} = \emptyset$ . It follows that

$$\mathbb{R}_+^{n+1} = \bigcup_{i \in \mathbb{Z}} \widehat{O}_i = \bigcup_{i \in \mathbb{Z}} \widehat{O}_i \setminus \widehat{O}_{i+1} = \bigcup_{i \in \mathbb{Z}} \bigcup_{j \in \mathbb{N}} T_{i,j},$$

where  $T_{i,j} := \widetilde{Q}_{i,j} \cap (\widehat{O}_i \setminus \widehat{O}_{i+1})$ . Thus,

$$f = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{N}} C_{(\Psi)} \int_0^\infty \Psi(t\sqrt{L}) \left( \chi_{T_{i,j}} t^2 L e^{-t^2 L} (f) \right) \frac{dt}{t} =: \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{N}} \lambda_{i,j} \alpha_{i,j}$$

converges in  $L^2(\mathbb{R}^n)$  due to the fact that  $f \in L^2(\mathbb{R}^n)$  (see [40, (3.11)]), where, for any  $i \in \mathbb{Z}$  and  $j \in \mathbb{N}$ ,  $\lambda_{i,j} := 2^i \|\chi_{Q_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$  and  $\alpha_{i,j} := L^M(b_{i,j})$  with

$$b_{i,j} := \frac{C_{(\Psi)}}{\lambda_{i,j}} \int_0^\infty \Psi(t\sqrt{L}) \left( \chi_{T_{i,j}} t^2 L e^{-t^2 L} (f) \right) \frac{dt}{t}.$$

By an argument similar to that used in [40, pp.477-479], we find that there exists a positive constant  $\widetilde{C}$  such that, for each  $i \in \mathbb{Z}$  and  $j \in \mathbb{N}$ ,  $\widetilde{C} \alpha_{i,j}$  is a  $(p(\cdot), \infty, M)_L$ -atom associated with the cube  $30Q_{i,j}$ . Moreover, by Lemma 2.9, Remark 2.8 and (3.15), we conclude that

$$\begin{aligned} & \mathcal{A}(\{\lambda_{i,j}\}_{j \in \mathbb{N}}, \{30Q_{i,j}\}_{j \in \mathbb{N}}) \\ & \lesssim \left\| \left\{ \sum_{i \in \mathbb{Z}, j \in \mathbb{N}} \left[ \frac{\lambda_{i,j} \chi_{Q_{i,j}}}{\|\chi_{Q_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{p_-} \right\}^{\frac{1}{p_-}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \left\| \left\{ \sum_{i \in \mathbb{Z}, j \in \mathbb{N}} [2^i \chi_{Q_{i,j}}]^{p_-} \right\}^{\frac{1}{p_-}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ & \sim \left\| \left\{ \sum_{i \in \mathbb{Z}} [2^i \chi_{O_i}]^{p_-} \right\}^{\frac{1}{p_-}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \left\| \left\{ \sum_{i \in \mathbb{Z}} [2^i \chi_{O_i \setminus O_{i+1}}]^{p_-} \right\}^{\frac{1}{p_-}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ & \sim \|\mathcal{N}_L^*(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H_{L, \max}^{p(\cdot)}(\mathbb{R}^n)} < \infty, \end{aligned}$$

which implies that  $f \in H_{L, \text{at}, M}^{p(\cdot), \infty}(\mathbb{R}^n)$  and hence (3.14) holds true. This finishes the proof of (3.13).

Finally, by Lemma 3.4 and the definitions of  $H_{L, \max}^{p(\cdot), \phi, a}(\mathbb{R}^n)$  and  $H_{L, \max}^{p(\cdot), \mathcal{F}}(\mathbb{R}^n)$ , we immediately find that

$$\left[ H_{L, \max}^{p(\cdot), \mathcal{F}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right] \subset \left[ H_{L, \max}^{p(\cdot), \phi, a}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right],$$



which, together with (3.8), (3.13) and Remark 1.7, implies that, for any  $q \in (1, \infty]$  and  $M \in (\frac{n}{2}[\frac{1}{p_-} - 1], \infty) \cap \mathbb{N}$ ,

$$\left[ H_{L, \text{at}, M}^{p(\cdot), q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right] = \left[ H_{L, \max}^{p(\cdot), \phi, a}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right] = \left[ H_{L, \max}^{p(\cdot), \mathcal{F}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right].$$

From this, Remark 1.14 and a density argument, we further deduce that the spaces  $H_{L, \text{at}, M}^{p(\cdot), q}(\mathbb{R}^n)$ ,  $H_{L, \max}^{p(\cdot), \phi, a}(\mathbb{R}^n)$  and  $H_{L, \max}^{p(\cdot), \mathcal{F}}(\mathbb{R}^n)$  coincide with equivalent quasi-norms, which completes the proof of Theorem 1.11.  $\square$

## 4 Proof of Theorem 1.17

In this section, we give the proof of Theorem 1.17, via beginning with establishing the following conclusion.

**Proposition 4.1.** *Let  $L$  be as in Theorem 1.17 and  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Then there exists a positive constant  $C$  such that, for all  $f \in L^2(\mathbb{R}^n)$ ,*

$$(4.1) \quad \|f_{L, \nabla}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f_{L, +}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

To prove Proposition 4.1, we need several auxiliary estimates.

**Lemma 4.2.** *Let  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$  and  $\lambda \in (n/p_-, \infty)$ . Then there exists a positive constant  $C$  such that, for any measurable function  $F$  on  $\mathbb{R}_+^{n+1}$ ,*

$$(4.2) \quad \|N_\lambda^1(F)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|M_1(F)\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where  $N_\lambda^1(F)$  and  $M_1(F)$  are as in (3.3), respectively, (3.1).

*Proof.* To prove this lemma, it suffices to show that, for all  $x \in \mathbb{R}^n$ ,

$$(4.3) \quad N_\lambda^1(F)(x) \leq \left\{ \mathcal{M} \left( [M_1(F)]^{n/\lambda} \right) (x) \right\}^{\lambda/n},$$

where  $\mathcal{M}$  denotes the Hardy-Littlewood maximal function defined in Remark 1.10. Indeed, if (4.3) is proved, then, by Remark 2.7 and the fact that  $\lambda \in (n/p_-, \infty)$ , we find that (4.2) holds true.

Next we show (4.3). By the definition of  $M_1(F)$ , we know that, for any  $t \in (0, \infty)$ ,  $x, y \in \mathbb{R}^n$  and  $z \in B(x - y, t)$ ,  $|F(x - y, t)| \leq M_1(F)(z)$ . From this and the fact that  $B(x - y, t) \subset B(x, |y| + t)$ , we deduce that

$$\begin{aligned} |F(x - y, t)|^{n/\lambda} &\leq \frac{1}{|B(x - y, t)|} \int_{B(x - y, t)} [M_1(F)(z)]^{n/\lambda} dz \\ &\leq \left( 1 + \frac{|y|}{t} \right)^n \mathcal{M} \left( [M_1(F)]^{n/\lambda} \right) (x), \end{aligned}$$

which further implies that (4.3) holds true. This finishes the proof of Lemma 4.2.  $\square$

For any  $\epsilon, N \in (0, \infty)$ ,  $f \in L^2(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ , let

$$f_{L,+}^{*,\epsilon,N}(x) := \sup_{t \in (0, \infty)} \left| e^{-t^2 L}(f)(x) \right| \frac{t^N}{[(t + \epsilon)(1 + \epsilon|x|)]^N},$$

$$(4.4) \quad f_{L,\nabla}^{*,\epsilon,N}(x) := \sup_{t \in (0, 1/\epsilon), |x-y| < t} \left| e^{-t^2 L}(f)(y) \right| \frac{t^N}{[(t + \epsilon)(1 + \epsilon|y|)]^N}$$

and, for all  $\lambda \in (0, \infty)$ ,

$$M_L^{\lambda,\epsilon,N}(f)(x) := \sup_{t \in (0, 1/\epsilon), y \in \mathbb{R}^n} \left| e^{-t^2 L}(f)(y) \right| \left( 1 + \frac{|x-y|}{t} \right)^{-\lambda} \frac{t^N}{[(t + \epsilon)(1 + \epsilon|y|)]^N}.$$

By an argument similar to that used in the proof of (4.3), we obtain the following conclusion, the details being omitted.

**Lemma 4.3.** *Let  $L$  be as in Theorem 1.17 and  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ . Suppose that  $\lambda \in (0, \infty)$  and  $\phi \in \mathcal{S}(\mathbb{R})$  is an even function with  $\phi(0) = 1$ . Then it holds true that, for all  $\epsilon, N \in (0, \infty)$ ,  $f \in L^2(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ ,*

$$M_L^{\lambda,\epsilon,N}(f)(x) \leq \left\{ \mathcal{M} \left( [f_{L,\nabla}^{*,\epsilon,N}]^{n/\lambda} \right) (x) \right\}^{\lambda/n}.$$

Moreover, we have the following lemma.

**Lemma 4.4.** *Let  $L$  be as in Theorem 1.17 and  $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ . For any  $\gamma, \lambda, \epsilon, N \in (0, \infty)$  and  $f \in L^2(\mathbb{R}^n)$ , let*

$$E := \left\{ x \in \mathbb{R}^n : M_L^{\lambda,\epsilon,N}(f)(x) \leq \gamma f_{L,\nabla}^{*,\epsilon,N}(x) \right\}.$$

*Then there exists a positive constant  $C$ , independent of  $\epsilon, N$  and  $f$ , such that, for all  $x \in E$ ,*

$$(4.5) \quad f_{L,\nabla}^{*,\epsilon,N}(x) \leq C \left\{ \mathcal{M} \left( [f_{L,+}^*]^{n/\lambda} \right) (x) \right\}^{\lambda/n}.$$

*Proof.* Let  $x$  be a given point of  $E \subset \mathbb{R}^n$ . Then, by the definition of  $f_{L,\nabla}^{*,\epsilon,N}(x)$ , we easily know that there exists  $(y_0, t_0) \in \mathbb{R}_+^{n+1}$  such that  $t_0 \in (0, 1/\epsilon)$ ,  $|x - y_0| < t_0$  and

$$(4.6) \quad f_{L,\nabla}^{*,\epsilon,N}(x) \leq 2 \left| e^{-t_0^2 L}(f)(y_0) \right| \frac{t_0^N}{[(t_0 + \epsilon)(1 + \epsilon|y_0|)]^N}.$$

We claim that, for any  $s \in (0, 1)$ ,  $r \in (0, \infty)$  and  $\tilde{x} \in B(y_0, rt_0)$ ,

$$(4.7) \quad I(x, \tilde{x}, y_0, r, t_0) := \left| e^{-t_0^2 L}(f)(\tilde{x}) - e^{-t_0^2 L}(f)(y_0) \right|$$

$$\lesssim r^{\mu s} M_L^{\lambda, \epsilon, N}(f)(x) \left[ \frac{t_0}{(t_0 + \epsilon)(1 + \epsilon|y_0|)} \right]^{-N},$$

where  $\mu$  is as in Assumption 1.15. If this claim holds true, then, by choosing  $r$  small enough, we find that, for any  $\tilde{x} \in B(y_0, rt_0)$ ,

$$\begin{aligned} \mathbf{I}(x, \tilde{x}, y_0, r, t_0) &\lesssim r^{\mu s} \gamma f_{L, \nabla}^{*, \epsilon, N}(x) \left[ \frac{t_0}{(t_0 + \epsilon)(1 + \epsilon|y_0|)} \right]^{-N} \\ &\leq \frac{1}{4} f_{L, \nabla}^{*, \epsilon, N}(x) \left[ \frac{t_0}{(t_0 + \epsilon)(1 + \epsilon|y_0|)} \right]^{-N}, \end{aligned}$$

which, combined with (4.6), implies that, for any  $\tilde{x} \in B(y_0, rt_0)$ ,

$$\left| e^{-t_0^2 L}(f)(\tilde{x}) \right| \geq \frac{1}{4} f_{L, \nabla}^{*, \epsilon, N}(x) \left[ \frac{t_0}{(t_0 + \epsilon)(1 + \epsilon|y_0|)} \right]^{-N} \geq \frac{1}{4} f_{L, \nabla}^{*, \epsilon, N}(x).$$

Therefore, for all  $x \in \mathbb{R}^n$ , we have

$$\begin{aligned} \left[ f_{L, \nabla}^{*, \epsilon, N}(x) \right]^{n/\lambda} &\lesssim \frac{1}{|B(y_0, rt)|} \int_{B(y_0, rt_0)} \left| e^{-t_0^2 L}(f)(\tilde{x}) \right|^{n/\lambda} d\tilde{x} \\ &\lesssim \left( \frac{1+r}{r} \right)^n \frac{1}{|B(x, (1+r)t_0)|} \int_{B(x, (1+r)t_0)} \left| e^{-t_0^2 L}(f)(\tilde{x}) \right|^{n/\lambda} d\tilde{x} \\ &\lesssim \mathcal{M} \left( [f_{L, +}^*]^{n/\lambda} \right)(x) \end{aligned}$$

with  $\mathcal{M}$  as in Remark 1.10, namely, (4.5) holds true.

To complete the proof of Lemma 4.4, it remains to show (4.7). By the semigroup property of  $\{e^{-tL}\}_{t>0}$ , we know that

$$\begin{aligned} (4.8) \quad \mathbf{I}(x, \tilde{x}, y_0, r, t_0) &= \left| \int_{\mathbb{R}^n} \left[ K_{t_0^2/2}(\tilde{x}, z) - K_{t_0^2/2}(y_0, z) \right] e^{-t_0^2 L/2}(f)(z) dz \right| \\ &\leq \mathbf{I}_0 + \sum_{k=3}^{\infty} \mathbf{I}_k, \end{aligned}$$

where

$$\mathbf{I}_0 := \int_{B(y_0, 4t_0)} \left| K_{t_0^2/2}(\tilde{x}, z) - K_{t_0^2/2}(y_0, z) \right| \left| e^{-t_0^2 L/2}(f)(z) \right| dz$$

and, for each  $k \in \{3, 4, \dots\}$ ,

$$\mathbf{I}_k := \int_{U_k(y_0, t_0)} \left| K_{t_0^2/2}(\tilde{x}, z) - K_{t_0^2/2}(y_0, z) \right| \left| e^{-t_0^2 L/2}(f)(z) \right| dz$$

with  $U_k(y_0, t_0) := B(y_0, 2^k t_0) \setminus B(y_0, 2^{k-1} t_0)$ .

Observe that, for any  $s \in (0, 1)$ , by Assumptions 1.2 and 1.3, we find that, for all  $z \in \mathbb{R}^n$ ,

$$(4.9) \quad \left| K_{t_0^2/2}(\tilde{x}, z) - K_{t_0^2/2}(y_0, z) \right| \lesssim \frac{1}{t_0^n} \left[ \frac{|\tilde{x} - y_0|}{t_0} \right]^{\mu s} \left[ e^{-\frac{|\tilde{x} - z|^2}{ct_0^2}} + e^{-\frac{|y_0 - z|^2}{ct_0^2}} \right]^{1-s},$$

where  $c$  and  $\mu$  are as in Assumptions 1.3, respectively, 1.15. By this and the fact that, for any  $z \in B(y_0, 4t_0)$ ,  $|x - z| \leq 5t_0$  and

$$\frac{1 + \epsilon|z|}{1 + \epsilon|y_0|} \leq \frac{1 + \epsilon|z - y_0| + \epsilon|y_0|}{1 + \epsilon|y_0|} \lesssim 1,$$

we find that

$$\begin{aligned} (4.10) \quad I_0 &\lesssim \int_{B(y_0, 4t_0)} \frac{1}{t_0^n} \left( \frac{|\tilde{x} - y_0|}{t_0} \right)^{\mu s} e^{-t_0^2 L/2}(f)(z) dz \\ &\lesssim \frac{r^{\mu s}}{t_0^n} M_L^{\lambda, \epsilon, N}(f)(x) \int_{B(y_0, 4t_0)} \left( 1 + \frac{|x - z|}{t_0} \right)^\lambda \frac{[(t_0 + \epsilon)(1 + \epsilon|z|)]^N}{t_0^N} dz \\ &\lesssim r^{\mu s} M_L^{\lambda, \epsilon, N}(f)(x) \frac{[(t_0 + \epsilon)(1 + \epsilon|y_0|)]^N}{t_0^N}. \end{aligned}$$

Next we deal with  $I_k$  for all  $k \in \{3, 4, \dots\}$ . Since  $|\tilde{x} - y_0| < t_0$ , it follows that, for any  $z \in U_k(y_0, t_0)$ ,  $|\tilde{x} - y_0| \leq |y_0 - z|/4$  and hence

$$|\tilde{x} - z| \geq |z - y_0| - |y_0 - \tilde{x}| > |y_0 - z|/2.$$

From this, (4.9) and the fact that, for any  $z \in U_k(y_0, t_0)$ ,

$$\frac{1 + \epsilon|z|}{1 + \epsilon|y_0|} \leq 1 + \frac{\epsilon|z - y_0|}{1 + \epsilon|y_0|} \lesssim 2^k,$$

we deduce that

$$\begin{aligned} (4.11) \quad I_k &\lesssim \frac{r^{\mu s}}{t_0^n} \int_{U_k(y_0, t_0)} e^{-\frac{|y_0 - z|^2}{2ct_0^2}(1-s)} \left| e^{-t_0^2 L/2}(f)(z) \right| dz \\ &\lesssim \frac{r^{\mu s}}{t_0^n} M_L^{\lambda, \epsilon, N}(f)(x) 2^{\lambda k} e^{-\beta 2^{2k}} \int_{U_k(y_0, t_0)} \frac{[(t_0 + \epsilon)(1 + \epsilon|z|)]^N}{t_0^N} dz \\ &\lesssim r^{\mu s} 2^{k(\lambda + n + N)} e^{-\beta 2^{2k}} M_L^{\lambda, \epsilon, N}(f)(x), \end{aligned}$$

where  $\beta$  is a positive constant depending on  $s$  and  $c$ .

Combining (4.8), (4.10) and (4.11), we conclude that (4.7) holds true. This finishes the proof of Lemma 4.4.  $\square$

We also need the following technical lemma.

**Lemma 4.5.** *Let  $\epsilon \in (0, 1)$  and  $f \in L^2(\mathbb{R}^n)$ .*

(i) *It holds true that there exists a positive constant  $N$ , depending on  $f$ , such that  $f_{L, \nabla}^{*, \epsilon, N} \in L^{p(\cdot)}(\mathbb{R}^n)$ .*

(ii) *If  $f_{L, \nabla}^* \in L^{p(\cdot)}(\mathbb{R}^n)$ , then  $f_{L, \nabla}^* \in L^{p(\cdot)}(\mathbb{R}^n)$ .*

*Proof.* We first prove (i). Let  $\varphi(x) := e^{-x^2/c}$  for all  $x \in \mathbb{R}^n$ , where  $c$  is as in Assumption 1.3. Then  $\varphi \in \mathcal{S}(\mathbb{R}^n)$  and, by Assumption 1.3, we know that, for all  $y \in \mathbb{R}^n$ ,

$$\left| e^{-t^2 L}(f)(y) \right| \lesssim \int_{\mathbb{R}^n} \frac{1}{t^n} e^{-\frac{|y-z|^2}{ct^2}} |f(z)| dz \sim \varphi_t * (|f|)(y),$$

where, for  $t \in (0, \infty)$ ,  $\varphi_t(\cdot) := t^{-n} \varphi(\frac{\cdot}{t})$ , which, combined with [25, Theorem 2.3.20], implies that there exist a positive constant  $C_{(f)}$  and integers  $m$  and  $l$ , depending on  $f$ , such that, for all  $y \in \mathbb{R}^n$ ,

$$\begin{aligned} \left| e^{-t^2 L}(f)(y) \right| &\leq C_{(f)} \sum_{\beta \in \mathbb{Z}_+^n, |\beta| \leq l} \sup_{z \in \mathbb{R}^n} (|y|^m + |z|^m) |(\partial^\beta \varphi_t)(z)| \\ &\leq C_{(f)} \frac{(1 + |y|)^m}{\min\{t^n, t^{n+l}\}} (1 + t^m) \sum_{\beta \in \mathbb{Z}_+^n, |\beta| \leq l} \sup_{z \in \mathbb{R}^n} (1 + |z/t|^m) |(\partial^\beta \varphi)(z/t)| \\ &\leq C_{(f)} (1 + \epsilon |y|)^m \epsilon^{-m} (1 + t^m) (t^{-n} + t^{-n-l}). \end{aligned}$$

From this, we further deduce that, for all  $t \in (0, 1/\epsilon)$  and  $|y - x| < t$ ,

$$\left| e^{-t^2 L}(f)(y) \right| \frac{t^N}{[(t + \epsilon)(1 + \epsilon|y|)]^N} \leq C_{(f)} \frac{1}{(1 + \epsilon|y|)^{N-m}} \frac{1 + \epsilon^{-m}}{\epsilon^{m+N/2}} \left( \epsilon^{n-N/2} + \epsilon^{n+l-N/2} \right),$$

where  $N$  is chosen large enough such that  $N > \max\{2(n+l), m + n/p_-\}$ , which, together with the fact that  $\epsilon \in (0, 1)$  and  $1 + \epsilon|y| \geq \frac{1}{2}(1 + \epsilon|x|)$ , implies that

$$f_{L, \nabla}^{*, \epsilon, N}(x) \leq C_{(f)} \frac{1}{(1 + \epsilon|x|)^{N-m}} \frac{1 + \epsilon^{-m}}{\epsilon^{m+N-n}}.$$

Observe that, for all  $x \in \mathbb{R}^n$ ,

$$(1 + \epsilon|x|)^{m-N} \leq \epsilon^{m-N} (1 + |x|)^{m-N} \lesssim \epsilon^{m-N} [\mathcal{M}(\chi_{B(0,1)})(x)]^{(N-m)/n},$$

where  $\mathcal{M}$  denotes the Hardy-Littlewood maximal function defined in Remark 1.10. By this and Remark 2.7, we conclude that

$$\left\| f_{L, \nabla}^{*, \epsilon, N} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \left\| \mathcal{M}(\chi_{B(0,1)}) \right\|_{L^{\frac{N-m}{n} p(\cdot)}}^{(N-m)/n} \lesssim \left\| \chi_{B(0,1)} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty,$$

where the implicit positive constants depend on  $f$ ,  $n$ ,  $N$  and  $\epsilon$ . Therefore,  $f_{L, \nabla}^{*, \epsilon, N} \in L^{p(\cdot)}(\mathbb{R}^n)$ .

Next, we show (ii). For any  $\lambda \in (n/p_-, \infty)$  and  $\gamma \in (0, \infty)$ , let  $E$  be as in Lemma 4.4. Then, by Lemma 4.3 and Remark 2.7, we conclude that

$$\left\| f_{L, \nabla}^{*, \epsilon, N} \chi_E \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \frac{1}{\gamma} \left\| M_L^{\lambda, \epsilon, N}(f) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \frac{C_1}{\gamma} \left\| f_{L, \nabla}^{*, \epsilon, N} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

which, combined with Remark 1.1, implies that

$$\left\| f_{L, \nabla}^{*, \epsilon, N} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_-} \leq \left\| f_{L, \nabla}^{*, \epsilon, N} \chi_E \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_-} + \left( \frac{C_1}{\gamma} \right)^{p_-} \left\| f_{L, \nabla}^{*, \epsilon, N} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_-}$$

with the positive constant  $C_1$  independent of  $f$ . By this, Lemma 4.5 and choosing  $\gamma := 2^{1/p-}C_1$ , we find that

$$\left\| f_{L,\nabla}^{*,\epsilon,N} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 2^{1/p-} \left\| f_{L,\nabla}^{*,\epsilon,N} \chi_E \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

From this, Lemma 4.4 and Remark 2.7, we deduce that

$$(4.12) \quad \left\| f_{L,\nabla}^{*,\epsilon,N} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \left\| \left\{ \mathcal{M} \left( [f_{L,+}^*]^{n/\lambda} \right) \right\}^{\lambda/n} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f_{L,+}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

with the implicit positive constants independent of  $\epsilon$ . Notice that, for any  $x \in \mathbb{R}^n$ ,

$$(4.13) \quad f_{L,\nabla}^{*,\epsilon,N}(x) \geq \frac{2^{-N}}{(1+\epsilon|x|)^N} \sup_{t \in (0,1/\epsilon)} \left( \frac{t}{t+\epsilon} \right)^N \sup_{|y-x|<t} \left| e^{-t^2L}(f)(y) \right|$$

and that the right hand side of (4.13) increases to  $2^{-N}f_{L,\nabla}^*(x)$  as  $\epsilon \rightarrow 0^+$ , namely,  $\epsilon \in (0, \infty)$  and  $\epsilon \rightarrow 0$ . Thus, it follows, from the Fatou lemma (see [13, Theorem 2.61]) and (4.12), that

$$(4.14) \quad \|f_{L,\nabla}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 2^N \liminf_{\epsilon \rightarrow 0^+} \|f_{L,\nabla}^{*,\epsilon,N}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f_{L,+}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

which implies that  $f_{L,\nabla}^* \in L^{p(\cdot)}(\mathbb{R}^n)$  and hence completes the proof of Lemma 4.5.  $\square$

**Remark 4.6.** Due to (4.14), Proposition 4.1 seems to be proved. However, this is not the case, since the implicit positive constant in (4.14) depends on  $N$  and hence on  $f$ , which is not allowed in Proposition 4.1.

Indeed, we prove Proposition 4.1 by an argument similar to that used in the proof (4.14) and the observation that, if  $\|f_{L,+}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)}$  is finite, then  $\|f_{L,\nabla}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)}$  is also finite.

For an even function  $\phi \in \mathcal{S}(\mathbb{R})$  with  $\phi(0) = 1$ , let, for any  $\lambda \in (0, \infty)$  and  $x \in \mathbb{R}^n$ ,

$$M_{L,\phi}^\lambda(f)(x) := N_\lambda^1(\phi(t\sqrt{L})(f)) = \sup_{t \in (0,\infty), y \in \mathbb{R}^n} \left| \phi(t\sqrt{L})(f)(y) \right| \left( 1 + \frac{|x-y|}{t} \right)^{-\lambda}.$$

Particularly, when  $\phi := e^{-|\cdot|^2}$ , we denote  $M_{L,\phi}^\lambda(f)$  simply by  $M_L^\lambda(f)$ .

*Proof of Proposition 4.1.* For any  $\lambda \in (n/p_-, \infty)$  and  $\gamma \in (0, \infty)$ , let

$$F := \left\{ x \in \mathbb{R}^n : M_L^\lambda(f)(x) \leq \gamma f_{L,\nabla}^*(x) \right\}.$$

Then, by Lemma 4.2, we find that

$$\|f_{L,\nabla}^* \chi_{F^c}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \frac{1}{\gamma} \|M_L^\lambda(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \frac{C_2}{\gamma} \|f_{L,\nabla}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where  $C_2$  is a positive constant independent of  $f$ . Notice that

$$\|f_{L,\nabla}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_-} \leq \|f_{L,\nabla}^* \chi_F\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_-} + \|f_{L,\nabla}^* \chi_{F^c}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_-}.$$

From this and Lemma 4.5(ii), together with choosing  $\gamma := 2^{1/p-}C_2$ , we deduce that

$$(4.15) \quad \|f_{L,\nabla}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 2^{1/p-} \|f_{L,\nabla}^* \chi_F\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

On the other hand, by an argument similar to that used in the proof of Lemma 4.4, we conclude that, for all  $x \in F$ ,

$$f_{L,\nabla}^*(x) \lesssim \left\{ \mathcal{M} \left( [f_{L,+}^*]^{n/\lambda} \right) (x) \right\}^{\lambda/n}$$

with  $\mathcal{M}$  as in Remark 1.10, which, combined with (4.15) and Remark 2.7, implies that (4.1) holds true. This finishes the proof of Proposition 4.1.  $\square$

We end this section by proving Theorem 1.17.

*Proof of Theorem 1.17.* To show Theorem 1.17, by Remark 1.14 and the definitions of  $H_{L,\max}^{p(\cdot)}(\mathbb{R}^n)$  and  $H_{L,\text{rad}}^{p(\cdot)}(\mathbb{R}^n)$ , we only need to prove that

$$(4.16) \quad \left[ H_{L,\text{rad}}^{p(\cdot)}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right] \subset \left[ H_{L,\max}^{p(\cdot)}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \right],$$

since the inverse inclusion is obvious.

Let  $f \in H_{L,\text{rad}}^{p(\cdot)}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ . Then, by Proposition 4.1, we find that

$$\|f\|_{H_{L,\max}^{p(\cdot)}(\mathbb{R}^n)} = \|f_{L,\nabla}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f_{L,+}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|f\|_{H_{L,\text{rad}}^{p(\cdot)}(\mathbb{R}^n)} < \infty,$$

which implies that  $f \in H_{L,\max}^{p(\cdot)}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$  and hence (4.16) holds true. This finishes the proof of Theorem 1.17.  $\square$

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